FAIR SOLUTIONS TO THE RANDOM ASSIGNMENT PROBLEM

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Abstract. We study the problem of assigning indivisible goods to individuals where each is to receive one good. To guarantee fairness in the absence of monetary compensation, we consider random assignments that individuals evaluate according to first order stochastic dominance (sd). In particular, we find that solutions that guarantee sd-no-envy (e.g. the Probabilistic Serial) are incompatible even with the weak sd-core from equal division. Solutions on the other hand that produce assignments in the strong sd-core from equal division (e.g. Hylland and Zeckhauser’s Walrasian Equilibria from Equal Incomes) are incompatible with the strong sd-equal-division-lower-bound. As an alternative, we present a solution, based on Walrasian equilibria, that is sd-efficient, in the weak sd-core from equal division and satisfies the strong sd-equal-division-lower-bound.

Keywords: Probabilistic Serial; Sd-efficiency; Sd-envy-free; Sd-core from equal division; Sd-equal-division-lower-bound

JEL codes: C70, D63

1. Introduction

In many allocation problems, we have to assign indivisible objects to individuals where each is to receive at most one. Public housing associations assign apartments to residents, school districts assign seats to students and childcare cooperatives assign chores to its members.

If fairness is understood as equity, the indivisibility of assigned objects will often render any eventual allocation unfair. In order to guarantee fairness at least from an ex-ante perspective, many theorists as well as practitioners have considered lotteries.1 While the design of such lotteries has received a lot of attention in recent years, most of the work concentrates on their efficiency and incentive properties (i.e. what are the incentives for participants to reveal their true preferences) – see for example [Erdil and Ergin, 2008], [Pathak and Sethuraman, 2011], [Abdulkadiroğlu et al., 2015]. In this paper, we try to complement the literature by taking a closer look at the original motivation for applying a lottery and ask “when is a lottery

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1For example, since 2010 Berlin assigns 30% of seats at overdemanded secondary schools through a lottery, see Basteck et al. [2015].
fair?". For this, we draw on the rich literature on fair allocation and adapt various equity criteria to random assignments.

Adherence to formal equity criteria may be particularly important when distributing publicly funded (or subsidised) private goods such as school seats, where no individual – or group of individuals – should be discriminated against. In the following, we focus on equity criteria that compare each individual’s assignment to the assignments of others or to the average over all assignments. In addition we consider variants of the core from equal division, which can be seen as a group equity criterion. Perhaps surprisingly, we find that all equity criteria are compatible with Pareto-efficiency, while (some) equity criteria for individuals are in conflict with (some) equity criteria for groups.

Since preferences over lotteries are often difficult to elicit, assignment mechanisms typically use individuals’ preferences over sure objects. For example school choice mechanisms typically ask students to submit a ranking of schools that they would like to attend. To extend these preferences over sure objects to preferences over lotteries, we will follow Bogomolnaia and Moulin [2001] and rely on first order stochastic dominance (sd). This extension can be seen as the most conservative possible extension, in the sense that an individual will sd-prefer one lottery over another only if she prefers it for any von Neumann Morgenstern utility function compatible with her preferences over sure objects.

The paper is organised as follows. In Section 2, we formally define the set of allocation problems under consideration. Section 3 lays out equity criteria; Section 4 describes which of these are satisfied by the most prominent existing solutions. Section 5 contains our main results – we find that some equity criteria are incompatible with each other so that there exists no solution that is able to satisfy all of them. To bridge this gap, we derive a new solution that satisfies a maximal number of equity criteria while ensuring Pareto efficiency.

2. Random Assignments

We consider the problem of allocating \( n \) objects \( a \in A \) among \( n \) individuals \( i \in I \). Each individual \( i \) is to receive one object and holds preferences over objects given by a weak order \( \succeq_i \). Let \( >_i \) and \( \sim_i \) denote the associated strict preference and indifference relation, respectively. A preference profile is denoted as \( \succeq = (\succeq_i)_{i \in I} \). We restrict preference profiles to cases of objective indifference, i.e. an individual may only be indifferent between objects, if every other individual is indifferent as well. Hence we may drop the index and denote the objective indifference relation as \( \sim \). Formally,

\[
\forall a, b \in A, i, j \in I : \quad a \sim_i b \iff a \sim_j b \iff a \sim b.
\]

We will refer to the tuple \((A, I, \succeq)\) as an assignment problem (of size \( n \)). Let \( p_{i,a} \) denote the probability that individual \( i \) is assigned object \( a \). An individual (random) assignment is a probability distribution over \( A \), i.e. a vector \( p_i = (p_{i,a})_{a \in A} \) such

3possibly a null-object
4Objects that everyone is indifferent between may be interpreted as multiple copies of the same object, such as for example multiple seats at the same school.
that $\sum_{a \in A} p_{i,a} = 1$. The set of probability distributions over $A$ is denoted $\Delta(A)$. 
A random assignment, $p = (p_i)_{i \in I}$, is a collection of individual assignments such that $\forall a \in A : \sum_{i \in I} p_{i,a} = 1$. A solution $S$ maps assignment problems to (sets of) random assignments.

In order to analyse random assignments and solutions, we extend individuals’ preferences over objects to preferences over individual assignments,\footnote{The Birkhoff–von Neumann-Theorem ensures that any random assignment can be represented as a convex combination of deterministic assignments where each individual receives one object.} using first order stochastic dominance (sd): define an individual’s weak upper contour set of $a$ as

$$U_i(a) = \{ b \in A \mid b \succ_i a \}$$

and write $p_i \succsd_i p_i$ if

$$\forall a \in A : \sum_{b \in U_i(a)} p_{i,a} \geq \sum_{b \in U_i(a)} p_{i,b}.$$ 

In words, an individual weakly prefers an individual assignment if it guarantees her a weakly higher chance of receiving her most preferred object(s) and a weakly higher chance of receiving the most or second most preferred object(s) and ... so on. If one of the inequalities is strict write $p_i >sd_i p_i$. Note that stochastic dominance induces only a partial order over assignments.

At times, we will also evaluate individual assignments according to a vector of weights $w_i = (w_{i,a})_{a \in A} \in \mathbb{R}^n$, where $w_i$ is said to be compatible with $z_i$ if

$$\forall a, b \in A : w_{i,a} > w_{i,b} \iff a >_i b.$$ 

Analogously, a collection of weight vectors $w = (w_i)_{i \in I}$ is compatible with preference profile $z$, if the same can be said for each component. The set of all such collections $w$ is denoted $W(z)$. In some contexts – in particular where a social planner is able to elicit them – $w_i$ may be interpreted as von Neumann–Morgenstern (vNM) utilities, associating an expected utility of $w_i \cdot p_i$ with each individual assignment $p_i$.\footnote{We abstract from consumption externalities, so preferences over random assignments only depend on the individual component.}

Alternatively, the weights might constitute a value judgement on behalf of a social planer, who tries to go beyond a reported preferences profile when choosing between different random assignments. For example, a school board might find that moving to a different random assignment where in expectation some additional $k$ students receive their first rather than their second choice school is preferable even as another $k$ students receive only their third rather than their second most preferred school. Inevitably, such decisions have to be made and making them with respect to fixed weight vectors may increase transparency and accountability.

Finally, a random assignment $p$ is sd-efficient\footnote{We use $w_i \cdot p_i$ to denote the inner product of $w_i$ and $p_i$.} unless there exists another assignment $\tilde{p}$ such that

$$\forall i \in I : \tilde{p}_i \succsd_i p_i \quad \text{and} \quad \exists i \in I : \tilde{p}_i >sd_i p_i.$$ 

It is weakly sd-efficient unless there exists $\tilde{p}$ with $\tilde{p}_i >sd_i p_i$, for all $i \in I$. A random assignment $p$ is efficient with respect to $w$ unless there exists another assignment $\tilde{p}$

\footnote{Bogomolnaia and Moulin [2001] introduced this concept as ordinal efficiency, to highlight the coarse informational underpinning of the preference relation $\succsd_i$.}
such that
\[ \forall i \in I : \quad w_i \cdot \tilde{p}_i \geq w_i \cdot p_i \quad \text{and} \quad \exists i \in I : \; w_i \cdot \tilde{p}_i > w_i \cdot p_i. \]

To familiarize ourself with the above definitions, observe that \( p \) is sd-efficient if there exists a collection of compatible weight vectors \( w \in W(z) \) such that \( p \) is efficient with respect to \( w \).\(^9\)

### 3. Equity Criteria

A minimal fairness requirement on random assignments demands equal treatment of equals: two individuals with identical preferences should receive the same amount of all objects that fall in the same indifference class.

**Definition 1.** Given an assignment problem \((A, I, z)\), a random assignment \( p \) satisfies **equal treatment of equals** if for all \( i, j \) in \( I \) we have
\[
\begin{align*}
\forall a \in A : \quad \sum_{b \sim a} p_{i,b} &= \sum_{b \sim a} p_{j,b}.
\end{align*}
\]

Note that where preferences are strict, this reduces to \( z_i = z_j \Rightarrow p_i = p_j \). Equitable treatment of individuals who differ in their preferences is harder to define. If one refrains from interpersonal comparisons of utility (as we do here) envy-freeness is arguably the most prominent such criterion. To check whether an allocation is envy-free, we need to compare individuals’ assignments - each individual should then prefer her own over anyone else’s assignment.

The criterion was introduced to economic theory by Tinbergen [1946] (p. 55 f.)\(^10\) and independently formulated in a dissertation by Foley [1967]. Both Tinbergen and Foley view envy-freeness as a principle of ‘macrojustice’ and compare individual positions that encompass most (if not all) aspects of individual well-being.\(^11\) In contrast to these two early proponents of the criterion, we will treat envy-freeness as a principle of ‘microjustice’, applicable to an isolated allocation problem and make no amends for any inequities that originate or persist outside of the model.

**Definition 2.** Given an assignment problem \((A, I, z)\), a random assignment \( p \) is **sd-envy-free** if for all \( i, j \) in \( I \) we have \( p_i \geq_p z_i \)\(^{sd} \) \( p_j \).

Observe that sd-envy-freeness implies equal treatment of equals: if two individuals \( i, j \) share the same preferences, sd-envy-freeness implies \( p_i \geq_p z_i^{sd} \) \( p_j \geq_p z_j^{sd} \) which is only possible if both \( i \) and \( j \) receive the same amount of all objects in the same indifference class.

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\(^9\)The converse holds as well, as proven (non-constructively) by McLennan [2002] and (constructively) by Manea [2008].

\(^10\)Tinbergen credits his professor, Dutch physicist Paul Ehrenfest, to have formulated the criterion in 1925 when they discussed the problem of interpersonal (non-)comparability.

\(^11\)Foley considers “material well-being” and includes not only private consumption goods and leisure but also local public goods. Tinbergen goes further and wants us to consider all of “life’s circumstances”, including for example health or social status. To make such comparisons viable, Tinbergen suggests to compare representative individuals of different social or occupational groups.

\(^12\)Bogomolnaia and Moulin [2001] are the first to formulate this property in the context of random assignments and refer to it simply as ‘envy-free’.
Sd-envy-freeness is satisfied whenever $p$ is envy-free with respect to all compatible weight vectors, i.e. if
\[ \forall w \in W(\mathbf{z}), \ i, j \in I : \ w_i \cdot p_i \geq w_i \cdot p_j. \]

From the perspective of a social planner who assumes that individuals evaluate random assignments in an expected utility framework, but who is informed only about their respective rankings over sure objects, sd-envy-freeness allows her to ensure envy-freeness with respect to individuals’ expected utilities despite her limited information on the latter.

Another natural yardstick to measure individuals’ assignments is equal division, denoted as $\left( \frac{1}{n} \right)$, i.e. the individual assignment that grants each object with probability $\frac{1}{n}$.

**Definition 3.** Given an assignment problem $(A, I, \mathbf{z})$, a random assignment $p$ satisfies
- the strong sd-equal-division-lower-bound if $\forall i \in I : \ p_i \geq \left( \frac{1}{n} \right)$,
- the weak sd-equal-division-lower-bound if $\exists i \in I : \ \left( \frac{1}{n} \right) > p_i$.

The weak notion is satisfied if the equal division lower bound is met for some collection of weight vectors $w \in W(\mathbf{z})$, while the strong notion requires that it is met for all such $w$. Hence, a social planner who only knows individual preferences over objects but chooses a random assignment that meets the strong sd-equal-division-lower-bound is able to ensure that the assignment also satisfies the equal-division-lower-bound with respect to individuals expected utilities, whatever they might be.

Observe that any random assignment that is sd-envy-free also meets the strong sd-equal-division-lower-bound: as each individual’s assignment stochastically dominates all other individual assignments, it also dominates the average of all individual assignments that is to say $\left( \frac{1}{n} \right)$. Formally:
\[ \forall i \in I, a \in A : \ \sum_{b \in U_i(a)} p_{i,b} \geq \left( \frac{1}{n} \right) \sum_{j \in I} \sum_{b \in U_j(a)} p_{j,b} = \sum_{b \in U_i(a)} \frac{1}{n}. \]

In addition, there are various equity criteria for groups of individuals. Such criteria might be especially important in allocating school seats and other publicly provided goods where we would want to ensure that no group – for example students of a particular neighbourhood or some demographic or ethnic group – is discriminated against and receives less than their ‘fair share’. Perhaps the most notable group equity criterion is the *core from equal division*.

**Definition 4.** Consider an assignment problem $(A, I, \mathbf{z})$. A group of individuals $G \subset I$ objects to a random assignment $\tilde{p}$ if there is an alternative assignment $p$ such that
- $\forall a \in A : \ \sum_{i \in G} p_{i,a} = \frac{|G|}{n}$ and
- $\forall i \in G : \ p_i \geq \tilde{p}_i$.

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the weak sd-core (from equal division).

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13I.e. for all $i \in I$, $w_i \cdot p_i \geq w_i \cdot (\frac{1}{n})$. 
The core from equal division extends the equal-division-lower-bound, allowing us to assess the assignments that a group of individuals receives with respect to the (aggregate) share that the group receives under equal division. In particular, a random assignment in the weak sd-core will satisfy the weak sd-equal-division-lower-bound, as can be easily verified by restricting attention to cases \( G = \{ i \} \) in Definition 4.

Observe that any element of the weak sd-core is weakly sd-efficient as any random assignment where everyone could be made strictly better off would be blocked by the grand coalition \( G = I \).

Still, the weak sd-core is comparatively large. Any objection by a blocking coalition would also entail a higher expected utility for each member. Thus, for any compatible weights \( w \in W(I) \), the associated weak \( w \)-core is included in the weak sd-core. Moreover, the weak sd-core is strictly larger than the union over all weak \( w \)-cores – see Appendix, Example 4.

To narrow down the weak sd-core, we consider a prominent subset – the strong sd-core – where coalitions can lean on indifferent members to formulate valid objections.

**Definition 5.** Consider an assignment problem \((A, I, \succ)\). A group of individuals \( G \subset I \) objects to a random assignment \( \tilde{p} \) if there is an alternative assignment \( p \) such that

- \( \forall a \in A : \sum_{i \in G} p_{i,a} = \frac{|G|}{n} \) and
- \( \forall i \in G : \; p_i \succ^d \tilde{p}_i \; \text{ and } \exists j \in G : \; p_j \succ^d \tilde{p}_j. \)

If there is no such objection that can be raised against a random assignment, the assignment is said to be in the *strong sd-core (from equal division)*.

A fortiori, a random assignment in the strong sd-core will satisfy the weak sd-equal-division-lower-bound. However, it may still violate the strong sd-equal-division-lower-bound – and equal treatment of equals:

**Example 1.** Consider three individuals where preferences of individual 1 and 2 are given as \( a \succ_1 b \succ_1 c \) while individual 3 prefers \( c \) over \( a \) and \( b \). Then the random assignment given by \( p_{1,a} = 1, p_{2,b} = 1 \) and \( p_{3,c} = 1 \) lies in the strong sd-core – individual 2 does not receive an assignment that sd-dominates equal division but an objection by \( G = \{ 2 \} \) is invalid, as \( \left( \frac{1}{n} \right) \not\succ^d 2 \; p_2 \). Also, there is no objection involving either individuals 1 or 3 who both receive their most preferred object, and could not be made as well off by any two-individual coalition. In a blocking coalition involving all three individuals, 1 and 3 would still have to receive their most preferred object, so that 2 could not be made better off. Thus, \( p \) is an element of the strong sd-core.

Conversely, equal division necessarily satisfies the strong sd-equal-division-lower-bound and equal treatment of equals, but will typically not be an element of the strong sd-core. In fact, any inefficient random assignment would be blocked by the grand coalition, so that all elements of the strong sd-core are sd-efficient.

\[ ^{14} \text{A random assignment } \tilde{p} \text{ lies in the } w \text{-core, unless there exist } G \subset I \text{ and assignment } p, \text{ such that for all } a \in A \text{ we have } \sum_{i \in G} p_{i,a} = \frac{|G|}{n} \text{ and for each member } i \text{ of } G \text{ we have } w_i \cdot p_i > w_i \cdot \tilde{p}_i. \]

\[ ^{15} \text{Nor would it constitute an element of the weak sd-core – consider the case } n = 2 \text{ where } a \succ_1 b \text{ and } b \succ_2 a. \]
Figure 1. Logical relations between equity criteria and 2 prominent solutions

--- Probabilistic Serial
----- Walrasian Equilibrium from Equal Incomes (Hylland and Zeckhauser)

Figure 1 provides a summary of all equity concepts discussed thus far, including their logical relations. In the center column, there are two independent and comparatively weak equity criteria. The weak sd-equal-division-lower-bound in particular can be strengthened in different ways by either allowing for group comparisons (see the right hand side) or by replacing ‘not strictly worse’ by a stronger ‘weakly better’ (left hand side). Also note that sd-envy-freeness implies all other individual equity criteria.

The absence of any connecting arrow(s) between two properties marks their logical independence. To be explicit,

- \((\text{strong sd-equal-division-lower-bound} + \text{strong sd-core}) \Rightarrow \text{equal treatment of equals}\) (see Appendix, Example 3).
- \(\text{equal treatment of equals} \Rightarrow \text{weak sd-equal-division-lower-bound}\).\(^{16}\)
- \(\text{sd-envy-freeness} \Rightarrow \text{weak sd-core} \) (follows from Proposition 1).
- \((\text{strong sd-core} + \text{equal treatment of equals}) \Rightarrow \text{sd-equal-division-lower-bound}\) (follows from Proposition 2).

4. PROMINENT SOLUTIONS

So far, we have discussed efficiency and equity criteria for particular assignment problems. Let us extend these criteria to solutions, i.e. set valued mappings, defined on a domain of assignment problems that map a particular assignment problem to a set of random assignments.

\(^{16}\)For example, consider the case \(n = 3\) where \(a >_{1,2} b >_{1,2} c\) and \(b >_3 a >_3 c\), individuals 1 and 2 receive the same assignment \(p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/2, 1/2, 0)\) and 3 receives object \(c\).
We say that a solution $S$ satisfies criterion $X$ (where $X$ could stand for sd-efficiency, equal treatment of equals, sd-envy-freeness etc.) if for any assignment problem $e$ in the domain of $S$, all random assignments in $S(e)$ satisfy $X$.

In the following, we consider three prominent solutions to random assignment problems, to see which of the criteria they satisfy. As we will see, all three solutions can be interpreted as taking equal division for a starting point and differ only in the manner in which trades towards the efficiency frontier are conducted.

4.1. **Random Serial Dictatorship.** A frequently used approach towards an equitable solution of assignment problems, is Random Serial Dictatorship (RSD). It requires us to order our $n$ individuals randomly (where all $n!$ orderings are equally likely). The first in line may then choose her most preferred object, while the second chooses the most preferred among the $n-1$ remaining objects. The third individual chooses among $n-2$ available objects and so it continues, until the last in line receives the last object available. From an ex-ante perspective – that is before we have decided on a particular ordering of individuals – this procedure generates a random assignment.

With respect to the equity criteria analysed in Section 3, let us first point out that a random assignment generated via RSD satisfies equal treatment of equals and meets the strong sd-equal-division-lower-bound: each individual has a chance of $\frac{k}{n}$ to be among the first $k$ individuals to choose, in which case she is guaranteed one of her $k$-most preferred objects. However, for some preference profiles, RSD falls short of sd-envy-freeness [Bogomolnaia and Moulin, 2001].

The main weakness of RSD lies in the fact that it fails to ensure even weak sd-efficiency [Bogomolnaia and Moulin, 2001]:

**Example 2.** Consider the case $n = 4$ with a preference profile where $a >_{1,2} b >_{1,2} c >_{1,2} d$ and $b >_{3,4} a >_{3,4} d >_{3,4} c$. Then RSD produces the following random assignment

<table>
<thead>
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<th>$a$</th>
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<tr>
<td>$p_1$:</td>
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<tr>
<td>$p_2$:</td>
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<td>$p_3$:</td>
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<tr>
<td>$p_4$:</td>
<td>(\frac{1}{12})</td>
<td>(\frac{5}{12})</td>
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which (from an ex-ante perspective) is Pareto inferior to the random assignment $\tilde{p}_{1,2} = (1/2, 0, 1/2, 0)$, $\tilde{p}_{3,4} = (0, 1/2, 0, 1/2)$.

Moreover, since RSD may return random assignments that are not weakly sd-efficient, it also returns random assignments that are not in the weak sd-core from

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17 If an individual is indifferent between multiple objects, this tie can be broken in any way – since under our assumption of objective indifference all others will similarly be indifferent between the same objects, her choice does not affect any individual that has to choose at a later stage.

18 Typically, once we have found a random assignment $p$, we need to construct a Birkhoff–von Neumann decomposition and represent $p$ as a convex combination of deterministic assignments – only then can we implement $p$ by taking a lottery over all elements of the decomposition. One of the practical advantages of RSD is that the randomization occurs in the very first step where we choose an ordering of individuals. Once this order is fixed, the algorithm returns a deterministic assignment, obviating any appeal to the Birkhoff–von Neumann-Theorem.
equal division. This contrasts with an alternative description of RSD by Abdulkadir-Diroğlu and Sönmez [1998], who characterize it on the domain of strict preferences as “core from random endowments”. More precisely, they consider random initial allocations of goods to individuals where each of the $n!$ deterministic allocations is equally likely. Given their endowments, individuals then trade towards the unique core allocation. From an ex-ante perspective, the convex combination of these core allocations coincides with the convex combination of allocations generated by a fixed pecking order described above.

4.2. Probabilistic Serial. One solution that overcome RSD’s lack of efficiency from an ex-ante perspective is the Probabilistic Serial (PS) mechanism, introduced by Bogomolnaia and Moulin [2001]. It generates random assignments via “simultaneous eating” where individuals accumulate probability shares, starting with their most preferred object until it is exhausted, before moving down to their second most preferred object and so on.\footnote{Bogomolnaia and Moulin [2001] consider the case of strict preferences. Their mechanism can be easily generalized to accommodate objective indifferences, i.e. multiple copies of objects – see for example Hashimoto et al. [2014].}

Not only does PS generate sd-efficient random assignments, it also ensures sd-envy-freeness [Bogomolnaia and Moulin, 2001]. However, as follows from our Proposition 1, these assignments will not in general lie in the weak sd-core from equal division.

Again, if we think of the core as the set of allocations that might be reached from an initial allocation through trade among individuals, this is in contrast with an alternative description of PS by Kesten [2009], who characterizes PS on the domain of strict preferences as “Top Trading Cycles from Equal Division”. For each individual $i$, her initial assignment $(\frac{1}{n})$ is managed by $n$ “pseudo-agents” $i_a$, $a \in A$. Each $i_a$ controls an initial probability share $\frac{1}{n}$ of object $a$ and shares $i$’s preferences over goods. In the first round, pseudo-agents $i_a$ will offer shares of $a$ in exchange for an equal share of the most preferred of $i$’s objects that are still available in the market. Wherever there is a double coincidence of wants, probability shares are exchanged and withdrawn from the market. Over time $i_a$ will have exchanged the whole of her initial share of object $a$, or she finds that object $a$ is the most preferred among all remaining objects. In both cases, $i_a$ exits the market. After at most $n$ steps, this trading algorithm terminates and the sum of probability shares acquired by $i$’s pseudo-agents is found to coincide with the individual assignment $p_i$ generated by PS.

4.3. Walrasian equilibrium from equal incomes. A third prominent solution is offered by Hylland and Zeckhauser [1979], who adapt the familiar concept of a Walrasian equilibrium from equal incomes (WEEI) to assignment problems.\footnote{Hylland and Zeckhauser [1979] also allow for differences in income, justified for example by the seniority of committee members that need to be assigned to tasks. In the spirit of our equity criteria identified in Section 3, we will concentrate on the case of equal incomes.} In contrast to our setting, individuals report vNM utilities $w_i$. Nevertheless, as maximisation of expected utilities implies maximisation with respect to stochastic dominance, we
find that their solution not only satisfies sd-efficiency, but also many of the equity criteria formulated in Section 3.

Formally, define the set of price vectors as 
\[ Q = \{ q = (q_a)_{a \in A} \in \mathbb{R}^n \mid \forall a \in A : q_a \geq 0 \}. \]

Individuals purchase probability shares, maximizing their expected utility \( w \cdot p \) subject to a constrained budget \( B \in \mathbb{R}^+ \) and the constraint \( \sum_A p_i = 1 \).

**Fact 1.** Hylland and Zeckhauser [1979]. Consider an assignment problem \((A, I, \succ)\).

For any collection of compatible weights \( w \in W(z) \), there exists a Walrasian equilibrium from equal incomes, i.e. a tuple \((p, q, B) \in \Delta(A)^n \times Q \times \mathbb{R}^+\) such that both

\[ \forall i \in I, \; q_i \in \Delta(A) : \quad q \cdot p \leq B \quad \text{and} \quad (w_i \cdot \tilde{p} > w_i \cdot p \Rightarrow q \cdot \tilde{p} > B) \]

(preference maximisation),

and \( \forall a \in A : \sum_{i \in I} p_i^a = 1 \) (feasibility)

Not surprisingly, such a WEEI will be efficient and in the strong core with respect to \( w \). Moreover, the associated random assignment will also be sd-efficient and an element of the strong sd-core from equal division: any trade (resp. objection) that would make everyone (resp. members of \( G \)) weakly better off with respect to first order stochastic dominance would also yield an increase in individuals expected utility. Similarly, preference maximisation and equal budgets guarantee envy-freeness with respect to \( w \), i.e. for all \( i, j \) we have \( w_i \cdot p_i \geq w_i \cdot p_j \).

One condition that is not automatically satisfied, is equal treatment of equals. If however, we chose \( w_i = w_j \) whenever \( z_i = z_j \), and constrain these individuals to consume the same probability shares (whenever they are indifferent and might choose different shares), this will guarantee equal treatment without violating preference maximisation or feasibility. Hence, using appropriately chosen weights \( w \), there exists a (sub)solution of WEEI’s that selects from the strong sd-core from equal division and satisfies equal treatment of equals.

However, in contrast to both RSD and PS, Hylland and Zeckhauser’s solution (and any sub-solution) will necessarily violate the strong sd-equal-division-lower-bound, at least for some preference profiles - see Proposition 2.

Figure 1 relates the two sd-efficient solutions discussed so far to the equity criteria that they satisfy.

### 5. Main Results

In light of Figure 1, we may ask whether there exists a solution that is able to satisfy all of our equity criteria. The following Proposition answers that question in the negative.

**Proposition 1.** For every \( n \geq 4 \) there exist assignment problems of size \( n \), for which no random assignment simultaneously satisfies sd-envy-freeness and lies in the weak sd-core from equal division.

**Proof of Proposition 1.** Consider an assignment problem \((A, I, \succ)\) of size \( n \geq 4 \), label objects as \( a, b, c, d \) and \( o_5, o_6, \ldots, o_n \), individuals as \( 1, 2, 3, \ldots, n \) and let their preferences over objects be given by the following rank-order lists:
Figure 2. Three Results on Fair Solutions

Possibility Result, Proposition 3

Impossibility Results, Proposition 1 and 2

1: b, a, c, d, o₅, o₆, ..., oₙ,
2: a, c, b, d, o₅, o₆, ..., oₙ,
3: a, b, d, c, o₅, o₆, ..., oₙ,
4: j: a, b, c, d, o₅, o₆, ..., oₙ, ∀j = 4, 5, ..., n.

Intuitively, preferences of individuals j ≥ 4 could be described as ‘mainstream preferences’ while in the preferences of the first 3 individuals there are reversals in the ranking of objects a, b, c, d that create opportunities for welfare improving trade.

We will proceed by analysing an arbitrary sd-envy-free random assignment p and show that it is no element of the weak sd-core as there exists a valid objection by G = {1, 2, 3} who are better of trading only amongst themselves. As p is assumed to be sd-envy-free, and all individuals agree on the ranking of alternatives o₅, o₆, ..., oₙ, we know that pᵢ,oₖ = 1/ₙ for all i ∈ I, k ≥ 5. As p also satisfies equal treatment of equals, we can express the individual assignment of individuals j ≥ 4 as

\[ p_j = (p_{j,a}, p_{j,b}, p_{j,c}, p_{j,d}, ..., p_{j,oₙ}) = (\frac{1}{n} + \alpha, \frac{1}{n} - \alpha + \beta, \frac{1}{n} - \beta + \gamma, \frac{1}{n} - \gamma, ..., \frac{1}{n}) \]

with \( \alpha, \beta, \gamma \geq 0 \). Sd-envy-freeness then implies that p takes the form

<table>
<thead>
<tr>
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<th>a</th>
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<th>c</th>
<th>d</th>
<th>o_k</th>
</tr>
</thead>
<tbody>
<tr>
<td>p₁</td>
<td>( \frac{1}{n} - (n-1)\alpha )</td>
<td>( \frac{1}{n} + (n-1)\alpha + \beta )</td>
<td>( \frac{1}{n} - \beta + \gamma )</td>
<td>( \frac{1}{n} - \gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>p₂</td>
<td>( \frac{1}{n} + \alpha )</td>
<td>( \frac{1}{n} - \alpha - (n-1)\beta )</td>
<td>( \frac{1}{n} + (n-1)\beta + \gamma )</td>
<td>( \frac{1}{n} - \gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>p₃</td>
<td>( \frac{1}{n} + \alpha )</td>
<td>( \frac{1}{n} - \alpha + \beta )</td>
<td>( \frac{1}{n} - \beta - (n-1)\gamma )</td>
<td>( \frac{1}{n} + (n-1)\gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>p₄</td>
<td>( \frac{1}{n} + \alpha )</td>
<td>( \frac{1}{n} - \alpha + \beta )</td>
<td>( \frac{1}{n} - \beta + \gamma )</td>
<td>( \frac{1}{n} - \gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
</tbody>
</table>

Individuals 2 and 3 agree with j ≥ 4 on the most preferred object and hence receive it with probability \( p_{2,a} = p_{3,a} = p_{j,a} = \frac{1}{n} + \alpha \). Individual 1 receives object a with remaining probability \( p_{1,a} = \frac{1}{n} - (n-1)\alpha \). Similarly, individuals i ≠ 2 agree on the weak upper contour set \( U_i(b) = \{a, b\} \) and hence receive a or b with probability
\( p_{1,a} + p_{1,b} = \frac{3}{n} + \beta \) - leaving individual 2 with the remaining probability \( p_{2,b} = \frac{1}{n} - \alpha - (n-1)\beta \). Finally, individuals \( i \neq 3 \) agree on the upper contour set \( U_i(c) = \{a,b,c\} \) and hence receive \( a \), \( b \) or \( c \) with probability \( p_{1,a} + p_{1,b} + p_{2,c} = \frac{3}{n} + \gamma \) - leaving individual 3 with the remaining probability \( p_{3,c} = \frac{1}{n} - \beta - (n-1)\gamma \). The entries \( p_{i,d} \) then follow from the condition \( \sum_{x \in A} p_{i,x} = 1 \).

As all entries are non-negative, we find three additional constraints on \( \alpha, \beta, \gamma \):

\[
\begin{align*}
(I) & \quad \alpha \leq \frac{1}{n(n-1)} \quad (\Leftrightarrow p_{1,a} = \frac{1}{n} - (n-1)\alpha \geq 0) \\
(II) & \quad \beta \leq \frac{1}{n(n-1)} - \frac{\alpha}{n-1} \quad (\Leftrightarrow p_{2,b} = \frac{1}{n} - \alpha - (n-1)\beta \geq 0) \\
(III) & \quad \gamma \leq \frac{1}{n(n-1)} - \frac{\beta}{n-1} \quad (\Leftrightarrow p_{3,c} = \frac{1}{n} - \beta - (n-1)\gamma \geq 0)
\end{align*}
\]

We claim that the following random assignment \( \tilde{p} \) constitutes a valid objection by group \( G = \{1,2,3\} \), who can do better by trading exclusively amongst themselves:

<table>
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<tr>
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<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>( o_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{p}_1 )</td>
<td>0</td>
<td>( \frac{3}{n} - \alpha - \beta )</td>
<td>( \alpha + \beta + \gamma )</td>
<td>( \frac{1}{n} - \gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>( \tilde{p}_2 )</td>
<td>( \frac{1}{n} + \alpha )</td>
<td>0</td>
<td>( \frac{3}{n} - \alpha - \beta - \gamma )</td>
<td>( \beta + \gamma )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>( \tilde{p}_3 )</td>
<td>( \frac{2}{n} - \alpha )</td>
<td>( \alpha + \beta )</td>
<td>0</td>
<td>( 2\frac{1}{n} - \beta )</td>
<td>( \frac{1}{n} )</td>
</tr>
<tr>
<td>( \tilde{p}_4 )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{1}{n} )</td>
<td>( \frac{1}{n} )</td>
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</table>

The random assignment is well defined, as all sums \( \sum_{x \in A} \tilde{p}_{i,x} = 1 = \sum_{i \in I} \tilde{p}_{i,x} \) and all entries are non-negative, given that \( \alpha, \beta, \gamma \leq \frac{1}{n} \) - see (I)-(III). Moreover, \( G \)'s resource constraint is met, as \( \sum_{i \in G} \tilde{p}_{i,x} = \frac{3}{n} \), for all \( x \in A \).

It remains to show that for all \( i \in G \), \( \tilde{p}_i \succ^i p_i \). First, consider individual 1. Here we find that she receives her most preferred object with strictly greater probability

\[
\tilde{p}_{1,b} - p_{1,b} = \frac{2}{n} - \frac{n\alpha - 2\beta}{n(n-1)} - \frac{1}{n-1} - \frac{2}{n} \geq \frac{n-4}{n(n-1)} \geq 0
\]

where (I) and (II) are used in the inequality. Moreover, she also receives her first or second object with greater probability than before:

\[
(\tilde{p}_{1,b} + \tilde{p}_{1,a}) - (p_{1,b} + p_{1,a}) = \frac{1}{n} - \alpha - \frac{2\beta}{n} > \frac{1}{n} - \frac{3}{n(n-1)} = \frac{n-4}{n(n-1)} \geq 0.
\]

As she receives her least preferred object \( d \) with the same probability as before \( (\tilde{p}_{1,d} = p_{1,d} = \frac{1}{n} - \gamma) \), we conclude that \( \tilde{p}_1 \succ^1 p_1 \).

Next, consider individual 2. She receives her most preferred object \( a \) with the same probability as before \( (\tilde{p}_{2,a} = p_{2,a} = \frac{1}{n} + \alpha) \) but receives her second most preferred object with higher probability:

\[
\tilde{p}_{2,c} - p_{2,c} = \frac{2}{n} - \alpha - \frac{n\beta - 2\gamma}{n(n-1)} - \frac{1}{n-1} + \frac{2\gamma}{n} \geq \frac{2}{n} - \frac{1}{n-1} - \frac{2}{n} \geq \frac{n-4}{n(n-1)} \geq 0,
\]

where we use (II) in the first and (III) in the second inequality. For the probability of receiving her least preferred object, we find (using (III))

\[
\tilde{p}_{2,d} - p_{2,d} = \beta - \frac{1}{n} < 0
\]
so that in conclusion $\tilde{p}_2 \triangleright^d p_3$. Finally, consider individual 3. Her most preferred object is $a$, which she now receives with strictly greater probability:

$$\tilde{p}_{3,a} - p_{3,a} = \frac{1}{n} - 2\alpha \geq \frac{1}{n} - \frac{2}{n(n-1)} = \frac{n-3}{n(n-1)} > 0.$$ 

As the probability of receiving one of her two most preferred objects remains unchanged ($\tilde{p}_{3,a} + \tilde{p}_{3,b} = p_{3,a} + p_{3,b} = \frac{2}{n} + \beta$) and as she now receives her least preferred object with zero probability, she too strictly prefers $\tilde{p}$ over $p$, rendering $\tilde{p}$ a valid objection by group $G = \{1, 2, 3\}$. □

According to Proposition 1, the Probabilistic Serial can be seen as a maximally fair solution with respect to our identified equity criteria - no other solution that is similarly sd-envy-free, can in addition select from the weak (or strong) sd-core from equal division.

That raises the question, whether there exist other maximally fair solutions – can we satisfy all remaining equity criteria once we give up sd-envy-freeness? Again, the answer is no.

**Proposition 2.** For every $n \geq 3$ there exist assignment problems of size $n$, for which no random assignment simultaneously satisfies the strong sd-equal-division-lower-bound and lies in the strong sd-core from equal division.

**Proof.** Consider the assignment problem $(A, I, \succ)$ where $I = \{1, 2, 3\}$ and preferences over $A$ are given as $a \succ_1 b \succ_1 c$ and $b \succ_3 a \succ_3 c$.

Any random assignment $p$ that satisfies the strong sd-equal-division-lower-bound will assign object $c$ with probabilities $p_{i,c} \leq \frac{1}{3}$. But then, $p_{i,c} = \frac{1}{3}$ and $p$ takes the form

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</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>$\frac{1}{3} + \alpha$</td>
<td>$\frac{1}{3} - \alpha$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$\frac{1}{3} + \beta$</td>
<td>$\frac{1}{3} - \beta$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$\frac{1}{3} - \alpha - \beta$</td>
<td>$\frac{1}{3} + \alpha + \beta$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq \frac{1}{3}$. For $p$ to lie in the strong sd-core it has to be sd-efficient, i.e. $\alpha + \beta = \frac{1}{3}$. Either $\alpha$ or $\beta$ will then be less than $\frac{1}{3}$ - assume w.l.o.g. that $\alpha < \frac{1}{3}$. But then, starting from equal division, individual 3 could exclusively trade with 1 and arrive at the following alternative random assignment $\tilde{p}$

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{p}_1$</td>
<td>$\frac{1}{3} + \alpha + \beta$</td>
<td>$\frac{1}{3} - \alpha - \beta$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\tilde{p}_2$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>$\tilde{p}_3$</td>
<td>$\frac{1}{3} - \alpha - \beta$</td>
<td>$\frac{1}{3} + \alpha + \beta$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

While this is a matter of indifference for individual 3, it is strictly preferred by 1. Thus, $\tilde{p}$ is a valid objection to $p$ by the group of individuals $\{1, 3\}$ and goes to show that $p$ is not in the strong sd-core from equal division. As $p$ was chosen as an arbitrary random assignment satisfying the strong sd-equal-division-lower-bound. The counterexample can be extended straightforwardly to the case $n > 3$ by choosing
preferences as \( a \succ i \ b \succ i \ c \succ i \ o_i \succ i \cdots \succ i \ o_n \) for \( i \in \{1, 2, \ldots, n-1\} \) and \( b \succ n \ a \succ n \ c \succ n \ o_k \succ n \cdots \succ n \ o_n \) .

Just as Proposition 1, Proposition 2 can be seen as a result on maximally fair solutions. In particular, (the sub-solution of) Hylland and Zeckhauser's Walrasian Equilibria from Equal Incomes, as described in Section 4.3, is maximally fair with respect to the identified equity criteria - no other solution that similarly selects from the strong sd-core can in addition satisfy the strong equal-division-lower-bound.

However, our two impossibility results leave space for a potential third class of maximally fair solutions. Can we give up on sd-envy-freeness and the restriction to the strong sd-core but satisfy all remaining equity criteria? Here, the answer is yes.

**Proposition 3.** For all \( n \) and all assignment problems \((A, I, \succ)\) of size \( n \), there exist random assignments that satisfy equal treatment of equals, meet the strong sd-equal-division-lower-bound, are in the weak sd-core from equal division and sd-efficient.

We will prove Proposition 3 by constructing a sequence of Walrasian equilibria with equal incomes. The limit of this sequence will then inherit many desirable properties, even if it is not itself a Walrasian equilibrium.

Our setting raises a number of problems for the existence of Walrasian equilibria. For one, individuals may be satiated and hence leave some of their income unspent, leading to a violation of Walras' law.

Second, if we restrict individuals' consumption sets to random assignments that meet the sd-equal-division-lower-bound, an equal division endowment lies on the boundary of individuals' consumption sets. Then, depending on the price vector, it may be that there is no random assignment that costs less than the initial endowment. This violates the so called strong survival assumption which is typically used to show that any quasi-equilibrium (whose existence may be establish more easily) is in fact a Walrasian equilibrium.

Third, the preference relation given by first order stochastic dominance is not continuous. For example an individual with preference \( a \succ i \ b \succ i \ c \) would (strictly) prefer \( p_i = (p_{i,a}, p_{i,b}, p_{i,c}) = (1/3, 2/3, 0) \) over \((1/3, 1/3, 1/3)\) but not over \((1/3 + \varepsilon, 1/3 - \varepsilon, 1/3)\).

To overcome the third problem, we will let individuals act as expected utility maximizers whose vNM utilities are compatible with their strict ordering of objects – preference maximization with respect to these vNM utilities then implies preference maximization with respect to first order stochastic dominance. The second problem can be overcome by relaxing consumption sets to be \( \varepsilon \)-close to the sd-equal-division-lower-bound – letting \( \varepsilon \) go to zero will then yield a limit allocation that satisfies all our desired criteria. To overcome the problem of satiated individuals, we have to allow for some 'slack' or a 'dividend' that increases the income of unsatiated individuals. In that, we follow Mas-Colell [1992]:

Let individuals’ consumption sets \( X_i \subset \mathbb{R}^n \) be closed, bounded and convex and let each individuals’ endowment \( y_i \) be in the interior of of \( X_i \). Let the set of possible price vectors be given as \( Q = \{ q = (q_a)_{a \in A} \in \mathbb{R}^n | |q| = \sum |q_a| \leq 1 \} \) and the state space be denoted as \( Z = X_1 \times X_2 \times \cdots \times X_n \times Q \). Individuals’ demand is guided by a (set-valued) preference map \( P_i: X_i \Rightarrow X_i \) and constrained by a budget \( q \cdot y_i + \frac{1-\varepsilon}{|q|} \) where the term
Fact 2. Theorem 1 in Mas-Colell [1992].

There exists a Walrasian equilibrium with slack, i.e. a state \( z = (x,q) \) such that

- \( \forall i \in I: \quad q \cdot x_i \leq q \cdot y_i + \frac{1-\|q\|}{\|q\|} \) and \( \bar{x}_i \in P_i(x_i) \implies q \cdot \bar{x}_i > q \cdot y_i + \frac{1-\|q\|}{\|q\|} \) (preference maximisation),

- \( \forall a \in A: \quad \sum_{i \in I} x_{i,a} = \sum_{i \in I} y_{i,a} \) (feasibility).

Proof of Proposition 3. Consider an assignment problem \((A,I,z)\) and a compatible collection of weight vectors \( w \in W(z) \). Define individuals’ consumption sets as

\[
X^*_i = \left\{ x_i = (x_{i,a})_{a \in A} \in \mathbb{R}^n \mid \sum_{a \in A} x_{i,a} \leq 1 + \varepsilon, \forall a \in A : x_{i,a} \geq 0 \text{ and } \sum_{b \in U_i(a)} x_{i,b} \geq \frac{|U_i(a)|}{n} - \varepsilon \right\}
\]

and endow each individual with a share of \( \frac{1}{n} \) of each object, i.e. \( y_{i,a} = \left( \frac{1}{n} \right) \). Note that consumption sets are closed, bounded and convex and that for \( \varepsilon > 0 \) endowments lie in the interior of the consumption set. Moreover, note that while for positive \( \varepsilon \) the consumption sets include bundles that cannot be interpreted as lotteries (\( \sum_A x_{i,a} \) may not be equal to 1), in the limit as we let \( \varepsilon \) go to zero, individuals are restricted to consume bundles that can be interpreted in this way and that (weakly) stochastically dominate \( \left( \frac{1}{n} \right) \). Let the set of possible price vectors be given as

\[
Q = \left\{ q = (q_a)_{a \in A} \in \mathbb{R}^n \mid \|q\| = \sum |q_a| \leq 1 \right\},
\]

and the state space as \( Z^e = X^*_1 \times X^*_2 \times \cdots \times X^*_n \times Q \). To provide individuals with continuous (strict) preferences, we define

\[
P_i: X^*_i \Rightarrow X^*_i : P_i(x_i) = \{ \bar{x}_i \in X^*_i \mid w_i \cdot \bar{x}_i > w_i \cdot x_i \}.
\]

Intuitively, under \( P_i \) a consumption bundle is preferred over another if it yields a higher expected utility with respect to vNM utilities \( w_i \) – except that bundles only approximate lotteries in that \( 1 - \varepsilon \leq \sum_A x_{i,a} \leq 1 + \varepsilon \). Clearly, \( P_i \) is irreflexive, convex-valued and has an open graph.

By Fact 2, for any \( \varepsilon \), there exists a Walrasian equilibrium with slack. Moreover, if we assume that all individuals with the same ordinal preferences, \( i \in G \), share the same weight vector \( w_i \), there exists a Walrasian equilibrium with \( x_i = x \) for all \( i \in G \) – for any other equilibrium, replacing individuals consumption bundles with

\[
x_i = \frac{\sum_G x'_{i,j}}{|G|}
\]

restores equal treatment of equals without violating preference maximisation or feasibility.

Consider a sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) with \( \varepsilon_k \downarrow 0 \) and a sequence of associated equilibria \( e_k = (x^k,q^k) \) satisfying equal treatment of equals. As the sequence of equilibria is bounded by \( X^*_1 \times X^*_2 \times \cdots \times X^*_n \times Q \), it has a convergent subsequence – and hence we
may assume w.l.o.g. that \( e^k \) is convergent itself. Denote the limit of that sequence as \( e^* = (x^*, q^*) \). Then \( x^* \) satisfies equal treatment of equals and, by construction of our consumption sets, is a random assignment that satisfies the strong sd-equal-division-lower-bound.

**Claim 1.** The random assignment \( x^* \) is in the weak sd-core from equal division.

*Proof of Claim:* Towards a contradiction, assume there exists a group \( G \subset I \) and another random assignment \( p \) such that \( \sum_{i \in G} p_i = |G| \left( \frac{1}{n} \right) \) and, for all \( i \in G \), \( p_i \succsd x_i^* \). The latter implies
\[
\forall i \in G, \varepsilon > 0 : \quad p_i \in X_i^\varepsilon,
\]
and for some sufficiently large \( \bar{k} \) we have
\[
\forall i \in G, k > \bar{k} : \quad w_i \cdot p_i > w_i \cdot x_i^{e^k}.
\]
Then by preference maximization we have
\[
\forall i \in G, k > \bar{k} : \quad q^{e^k} \cdot p_i > q^{e^k} \cdot \left( \frac{1}{n} \right) + \frac{1 - \|q^{e^k}\|}{\|q^{e^k}\|} \geq q^{e^k} \cdot \left( \frac{1}{n} \right).
\]
But this contradicts \( \sum_G q^{e^k} \cdot p_i = q^{e^k} \sum_G p_i = q^{e^k} |G| \left( \frac{1}{n} \right) \).

\[\blacksquare\]

**Claim 2.** The random assignment \( x^* \) is sd-efficient.

*Proof of Claim:* Towards a contradiction, assume there exists another random assignment \( p \) and a group \( G \subset I \) such that \( p_i \succsd x_i^* \) for all \( i \in G \) and \( \sum_G (p_i - x_i^*) = 0 \). As the trade \( (p_i - x_i^*) \) sd-improves individual assignments, we know that
\[
\forall i \in G : \quad w_i \cdot (p_i - x_i^*) > 0.
\]
Nevertheless, the bundle \( x_i^* + (p_i - x_i^*) \) may not be in the consumption set \( X_i^\varepsilon \) if there is a \( \alpha \) such that \( x_i^\varepsilon + (p_i - x_i^*) < 0 \). Only as \( x_i^\varepsilon \) approaches \( x_i^* \), a (scaled down) trade can always be executed:
\[
\exists \bar{k} : \quad \forall i \in G, k > \bar{k} : \quad x_i^{e^k} + \frac{1}{2} (p_i - x_i^*) \in X_i^{\varepsilon^k}.
\]
Then by preference maximization we have
\[
\forall i \in G, k > \bar{k} : \quad q^{e^k} \cdot \frac{1}{2} (p_i - x_i^*) > 0.
\]
But this contradicts \( \sum_G (p_i - x_i^*) = 0 \).

This completes the proof. \[\blacksquare\]

6. **Concluding Remarks**

We end this paper with two remarks. First, our impossibility results Proposition 1 and 2 illuminate the difference between two of the most prominent efficient solutions to the random assignment problem, namely the Probabilistic Serial (which is sd-envy-free) and Hylland and Zeckhauser’s WEEI (which selects from the strong sd-core) and show that no assignment mechanism may satisfy all equity criteria satisfied by either of the two. Such impossibility results may also be of practical importance where an assignment mechanism is challenged in court, for example by individuals who are unsatisfied with their eventual assignment. Here, claimants may
argue against an assignment mechanism by identifying specific equity criteria that have been violated. To judge the validity of such arguments, we would have to know whether the identified equity criteria are at least feasible – if that is not the case, the violation cannot serve as an argument for rejecting the assignment mechanism or the specific assignment that it produced.

Second, so far we have not considered the issue of strategic reports by individuals. Bogomolnaia and Moulin [2001] show that no assignment mechanism satisfies sd-efficiency, equal treatment of equals and strategy proofness. However for the Probabilistic Serial Kojima and Manea [2010] show that in large markets, with many individuals and sufficiently many copies of each object, reporting truthfully becomes a weakly dominant strategy. Hylland and Zeckhauser argue that individuals can only gain from misrepresenting their preferences if that influences prices and that in large markets “no individual can have a foreseeable effect on price[s]” [Hylland and Zeckhauser, 1979]. Our solution also rests on Walrasian equilibria, where it is rather demanding for individuals to foresee the effect that any misreport would have on prices. Hence, at least in large markets, our solution is similarly hard to manipulate. Moreover, our Proposition 3 raises another question: can we drop the sd-efficiency requirement and arrive at a strategy proof assignment mechanism that satisfies the strong sd-equal-division-lower-bound, equal treatment of equals and selects from the weak sd-core? The latter would ensure weak sd-efficiency and hence could be seen as improving upon Random Serial Dictatorship both in terms of fairness and efficiency.

**Example 3.** The following example shows that a random assignment in the strong sd-core from equal division that satisfies the strong sd-equal-division-lower-bound may nevertheless violate equal treatment of equals. Suppose individuals \(i \in \{1, 2, 3\}\) hold preferences \(a >_i b >_i c >_i d\) while individual 4 prefers object \(d\). Then in the following random assignment \(p\)

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<tbody>
<tr>
<td>(p_1):</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{4})</td>
<td>0</td>
</tr>
<tr>
<td>(p_2):</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{2})</td>
<td>(\frac{1}{4})</td>
<td>0</td>
</tr>
<tr>
<td>(p_3):</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>0</td>
</tr>
<tr>
<td>(p_4):</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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</table>

each individual receives an assignment \(p_i\) that stochastically dominates equal division, viz. \(p\) satisfies the strong sd-equal-division-lower-bound.

To see that \(p\) lies in the strong sd-core from equal division, observe first that it is sd-efficient - there exists no Pareto improving trade involving 4 (who already receives \(d\) with certainty) and no Pareto improving trade among 1, 2 and 3 (who hold identical preferences). Thus, the grand coalition will not object to \(p\). Next, consider objections by groups of size \(k < 4\). Individual 4 cannot support such an objection, as she would receive probability shares of her most preferred object of
Nor could the remaining individuals form a blocking coalition where everyone is (weakly) better off in a stochastic dominance sense, as someone would have to accept \( p_{i,d} > 0 \). Hence, \( p \) lies in the strong sd-core.

However, \( p \) does not satisfy equal treatment of equals, as \( p_1 = p_2 \neq p_3 \).

**Example 4.** The following example shows that the weak sd-core may include allocations that are not member of any weak \( w \)-core. Consider the case \( n = 4 \) and suppose that preferences of individual \( i \in \{1, 2\} \) are given as \( a >_i b >_i c >_i d \). The third individuals preferences are given as \( b >_3 a >_3 c >_3 d \) while the fourth individuals holds preferences \( a >_4 c >_4 b >_4 d \). Then the following random assignment \( p \)

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<tbody>
<tr>
<td>( p_1 ):</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
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<td>( p_2 ):</td>
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<td>( p_3 ):</td>
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<tr>
<td>( p_4 ):</td>
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lies in the weak sd-core from equal division: Consider an objection \( \tilde{p} \) by a blocking coalition \( G \) that includes 4. Since everyone agrees that \( d \) is the worst, we have \( \tilde{p}_{i,d} = \frac{1}{4} = p_{i,d} \) for all \( i \in G \). Moreover, since for \( 1, 2 \in G \) they would receive at least \( \frac{1}{4} \) of object \( a \) under \( \tilde{p} \), we have \( \tilde{p}_{4,a} = \frac{1}{2} \). But then \( \tilde{p}_4 >^{sd} p_4 \) is impossible, as 4 cannot receive less than 0 of \( b \).

Next, consider an objection \( \tilde{p} \) by a blocking coalition excluding 4. Since individuals 1, 2 and 3 agree that \( c \) is the third- and \( d \) is the forth-most preferred object, no one individual may receive more than \( \frac{1}{4} \) of \( d \) and more than \( \frac{1}{2} \) of \( c \) and \( d \). Thus we have \( \tilde{p}_{i,c} = \tilde{p}_{i,d} = \frac{1}{4} = p_{i,c} = p_{i,d} \) for all \( i \in G \). But then \( \tilde{p}_3 >^{sd} p_3 \) is impossible, as 3 receives his most preferred object \( b \) with maximal probability. Hence, finally, a blocking coalition \( G \) may only include individuals 1 and 2 – but since they have identical preferences, there is no scope for a valid objection that improves upon equal division.

However, for any compatible profile of vNM-utility functions the following allocation constitutes a valid objection by individuals 1,2, and 3 - provided \( \varepsilon \) is chosen sufficiently small:

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<tr>
<td>( p_1 ):</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{8} - \varepsilon )</td>
<td>( \frac{1}{4} + \varepsilon )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( p_2 ):</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{1}{8} - \varepsilon )</td>
<td>( \frac{1}{4} + \varepsilon )</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>( p_3 ):</td>
<td>0</td>
<td>( \frac{1}{2} + 2\varepsilon )</td>
<td>( \frac{1}{4} - 2\varepsilon )</td>
<td>( \frac{1}{4} )</td>
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**References**


