Dynamic Games: Numerical Methods and Applications

Day 1

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What are Dynamic Games?

- A tool for analyzing dynamic strategic interactions.
  - dynamic $\rightarrow$ forward-looking players optimize over time;
  - strategic $\rightarrow$ each player recognizes that its actions impact other players.

- Often used to track evolution of oligopolistic industries.
  - oligopolistic $\rightarrow$ neither perfectly competitive nor monopolistically competitive.

Dynamics + Strategic Interactions = Dynamic Games

- Combine literature on long-run industry equilibrium (Jovanovic 1982, Hopenhayn 1992, Melitz 2003) with game theory (Tirole 1988, Fudenberg & Tirole 1991).

Why use Dynamic Games?

- Key findings of empirical literature on industry evolution (Mueller 1986, Dunne, Roberts, & Samuelson 1988, Davis & Haltiwanger 1992):
  - Entry and exit occur simultaneously.
  - Heterogeneity among firms evolves endogenously in response to random occurrences.
  - Heterogeneity among firms persists over long stretches of time.

Why use Dynamic Games?

- Game theory revolution in economics: emphasis on analytically tractable models.
  - End effects.
  - Transitional dynamics.
  - Inherently dynamic phenomena.

Agenda

- From dynamic programming to dynamic games.
- Application: Quality ladder model without entry/exit.
- Markov-perfect industry dynamics.
- Existence, purification, and multiplicity of equilibrium.
- Application: Quality ladder model with entry/exit.

From Dynamic Programming...

- Time is discrete. The horizon is infinite.
- The state space $\Omega = \{1, 2, \ldots, L\}$ is finite.
- The state in period t is $\omega_t \in \Omega$. The law of motion is a controlled discrete-time, finite-state, first-order Markov process, where
$$\Pr(\omega_{t+1}|\omega_t, x_t)$$
is the probability that the state transits from ω_t to ω_{t+1} if the control is $x_t \in D(\omega_t)$ and $D(\omega_t)$ is the nonempty set of feasible controls in state ω_t.
- The objective is to maximize the expected NPV of payoffs
$$E\left\{\sum_{t=0}^{\infty} \beta^t \pi(\omega_t, x_t)\right\},$$
where $\beta \in [0,1)$ is the discount factor and $\pi(\omega_t, x_t)$ is the per-period payoff in state ω_t if the control is x_t.
- The value function $V(\omega)$ is the maximum expected NPV of present and future payoffs if the current state is ω. It satisfies the Bellman equation
$$V(\omega) = \max_{x \in D(\omega)} \pi(\omega, x) + \beta \sum_{\omega'=1}^{L} V(\omega')\Pr(\omega'|\omega, x) \tag{1}$$
and the optimal policy function $X(\omega)$ satisfies
$$X(\omega) \in \arg\max_{x \in D(\omega)} \pi(\omega, x) + \beta \sum_{\omega'=1}^{L} V(\omega')\Pr(\omega'|\omega, x).$$
- The collection of equation (1) for all states $\omega \in \Omega$ defines a system of nonlinear equations. The contraction mapping theorem ensures existence and uniqueness of a solution.

. . . to Dynamic Games

- N players.

- The law of motion is a controlled discrete-time, finite-state, first-order Markov process, where

$$\text{Pr}(\omega_{t+1}|\omega_t, x_t)$$

is the probability that the state transits from ω_t to ω_{t+1} if the controls are $x_t = (x_{1t}, \ldots, x_{Nt}) \in \times_{n=1}^{N} D_n(\omega_t)$ and $D_n(\omega_t)$ is the nonempty set of feasible controls of player n in state ω_t.

- $\pi_n(\omega_t, x_t)$ is the per-period payoff of player n in state ω_t if the controls are x_t.

- The value function $V_n(\omega)$ of player n satisfies the Bellman equation

$$V_n(\omega) = \max_{x_n \in D_n(\omega)} \pi_n(\omega, x_n, X_{-n}(\omega)) + \beta \sum_{\omega'=1}^{L} V_n(\omega')\text{Pr}(\omega'|\omega, x_n, X_{-n}(\omega)) \qquad (2)$$

and his optimal policy function $X_n(\omega)$ satisfies

$$X_n(\omega) \in \text{arg} \max_{x_n \in D_n(\omega)} \pi_n(\omega, x_n, X_{-n}(\omega)) + \beta \sum_{\omega'=1}^{L} V_n(\omega')\text{Pr}(\omega'|\omega, x_n, X_{-n}(\omega)). \qquad (3)$$

- The collection of equations (2) and (3) for all states $\omega \in \Omega$ and all players $n = 1, \ldots, N$ defines a Markov-perfect equilibrium. The contraction mapping theorem does not apply and neither existence nor uniqueness of a MPE is guaranteed.

... to Dynamic Games

- Special case: ω is a vector partitioned into
$$(\omega_1, \ldots, \omega_N),$$
where ω_n denotes the (one or more) coordinates of the state that describe player n.
Examples: Production capacity, marginal cost, product quality.
Nomenclature:
  - $\omega_n \in \Omega_n = \{1, 2, \ldots, L_n\}$ is the state of player n;
  - $\omega \in \times_{n=1}^{N} \Omega_n$ is the state of the game.

  Equations (2) and (3) can be written as
$$V_n(\omega) = \max_{x_n \in D_n(\omega)} \pi_n(\omega, x_n, X_{-n}(\omega)) + \beta \sum_{\omega'_1=1}^{L_1} \ldots \sum_{\omega'_N=1}^{L_N} V_n(\omega') \mathrm{Pr}(\omega'|\omega, x_n, X_{-n}(\omega)),$$
$$X_n(\omega) \in \arg\max_{x_n \in D_n(\omega)} \pi_n(\omega, x_n, X_{-n}(\omega)) + \beta \sum_{\omega'_1=1}^{L_1} \ldots \sum_{\omega'_N=1}^{L_N} V_n(\omega') \mathrm{Pr}(\omega'|\omega, x_n, X_{-n}(\omega)).$$

- Even more special case: Transitions in player n's state are controlled by player n's actions and are independent of the actions of other players and transitions in their states, i.e.,
$$\mathrm{Pr}\left(\omega'|\omega, x\right) = \prod_{n=1}^{N} \mathrm{Pr}_n\left(\omega'_n|\omega_n, x_n\right).$$

Quality Ladder Model without Entry/Exit

- Pakes, A. & McGuire, P. (1994) "Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model."

- Borkovsky, R., Doraszelski, U. & Kryukov, Y. (2010) "A User's Guide to Solving Dynamic Stochastic Games Using the Homotopy Continuation Method."

- Discrete time, infinite horizon.

- Two firms with potentially different product qualities

$$\omega = (\omega_1, \omega_2) \in \{1, \ldots, L\}^2 = \Omega.$$

- In each period, the timing is as follows:
  - Firms choose investments in quality improvements.
  - Product market competition takes place.
  - Investment outcomes and depreciation shocks are realized.

Product Market Competition

- Firm n's demand is

$$D_n(p_1, p_2; \omega) = M \frac{\exp\left(g(\omega_n) - p_n\right)}{1 + \sum_{k=1}^{2} \exp\left(g(\omega_k) - p_k\right)},$$

  where $M > 0$ is market size and

$$g(\omega_n) = \begin{cases} 3\omega_n - 4 & \text{if} \quad \omega_n \leq 5, \\ 12 + \ln\left(2 - \exp\left(16 - 3\omega_n\right)\right) & \text{if} \quad \omega_n > 5 \end{cases}$$

  maps product quality into consumers' valuations.

- Firm n solves

$$\max_{p_n \geq 0} D_n(p_1, p_2; \omega)(p_n - c),$$

  where c is marginal cost of production.

- FOC:

$$0 = 1 - \frac{1 + \exp\left(g(\omega_{-n}) - p_{-n}\right)}{1 + \exp\left(g(\omega_n) - p_n\right) + \exp\left(g(\omega_{-n}) - p_{-n}\right)}(p_n - c), \quad n \neq -n.$$

- Compute Nash equilibrium $(p_1(\omega), p_2(\omega))$ by solving system of FOCs.

- Firm n's profit is

$$\pi_n(\omega) = D_n(p_1(\omega), p_2(\omega); \omega)(p_n(\omega) - c).$$

Investment Dynamics

- Let $x_n \geq 0$ be firm n's investment in quality improvements.
- Law of motion:
  - Successful investment has probability $\frac{\alpha x_n}{1+\alpha x_n}$.
  - Depreciation shock has probability δ.
- Transition probability: If $\omega_n \in \{2, \ldots, L-1\}$, then

$$\Pr(\omega_n' | \omega_n, x_n) = \begin{cases} \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n + 1, \\ \frac{1-\delta+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n, \\ \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n - 1. \end{cases}$$

  If $\omega_n \in \{1, L\}$, then

$$\Pr(\omega_n' | 1, x_n) = \begin{cases} \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = 2, \\ \frac{1+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = 1, \end{cases}$$

$$\Pr(\omega_n' | L, x_n) = \begin{cases} \frac{1-\delta+\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = L, \\ \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = L - 1. \end{cases}$$

Bellman Equation

- Let $V_n(\omega)$ denote the expected NPV to firm n if the current state is ω.

- Firm n's Bellman equation is

$$V_n(\omega) = \max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega'_n=1}^{L} W_n(\omega'_n; \omega_{-n}, x_{-n}(\omega)) \Pr(\omega'_n | \omega_n, x_n),$$

where

  - the expectation (with respect to its rival's successor state) of firm n's continuation value in state ω'_n is

$$W_n(\omega'_n; \omega_{-n}, x_{-n}(\omega)) = \sum_{\omega'_{-n}=1}^{L} V_n(\omega') \Pr(\omega'_{-n} | \omega_{-n}, x_{-n}(\omega));$$

  - $x_{-n}(\omega)$ is the rival's investment strategy;

  - $\beta \in [0, 1)$ is the discount factor.

Investment Strategy

- Firm n's investment strategy is

$$x_n(\omega) = \arg\max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega'_n=1}^{L} W_n(\omega'_n)\text{Pr}(\omega'_n|\omega_n, x_n),$$

where $W_n(\omega'_n)$ is shorthand for $W_n(\omega'_n; \omega_{-n}, x_{-n}(\omega))$.

- If $\omega_n \in \{2, \ldots, L-1\}$, then

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha\left((1-\delta)(W_n(\omega_n+1) - W_n(\omega_n)) + \delta(W_n(\omega_n) - W_n(\omega_n-1))\right)\}}}{\alpha}.$$

If $\omega_n \in \{1, L\}$, then

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha(1-\delta)\left(W_n(2) - W_n(1)\right)\}}}{\alpha},$$

$$x_n(\omega) = \frac{-1 + \sqrt{\max\{1, \beta\alpha\delta\left(W_n(L) - W_n(L-1)\right)\}}}{\alpha}.$$

Equilibrium

- Profits from product market competition are symmetric:

$$\pi_1(\omega_1, \omega_2) = \pi_2(\omega_2, \omega_1).$$

  The remaining primitives are also symmetric.

- Symmetric Markov perfect equilibrium (MPE):
  - Value function $V_1(\omega_1, \omega_2) = V(\omega_1, \omega_2)$ and $V_2(\omega_1, \omega_2) = V(\omega_2, \omega_1)$.
  - Policy function $x_1(\omega_1, \omega_2) = x(\omega_1, \omega_2)$ and $x_2(\omega_1, \omega_2) = x(\omega_2, \omega_1)$.

- Existence in pure strategies is guaranteed (Doraszelski & Satterthwaite 2010), uniqueness is not.

- The goal is to compute the value and policy functions (or, more precisely, $L \times L$ matrices) $\mathbf{V}$ and $\mathbf{x}$.

Computation: Pakes & McGuire (1994) Algorithm

1. Make initial guesses $\mathbf{V}^0$ and $\mathbf{x}^0$, choose a stopping criterion $\epsilon > 0$, and initialize the iteration counter to $k = 1$.

2. For all states $\omega \in \Omega$ compute

$$x^{k+1}(\omega) = \arg\max_{x_1 \geq 0} \pi_1(\omega) - x_1 + \beta \sum_{\omega_1'=1}^{L} W^k(\omega_1') \Pr(\omega_1' | \omega_1, x_1)$$

   and

$$V^{k+1}(\omega) = \pi_1(\omega) - x^{k+1}(\omega) + \beta \sum_{\omega_1'=1}^{L} W^k(\omega_1') \Pr(\omega_1' | \omega_1, x^{k+1}(\omega)),$$

   where

$$W^k(\omega_1') = \sum_{\omega_2'=1}^{L} V^k(\omega') \Pr(\omega_2' | \omega_2, x^k(\omega_2, \omega_1)).$$

3. If

$$\max_{\omega \in \Omega} \left| \frac{V^{k+1}(\omega) - V^k(\omega)}{1 + |V^{k+1}(\omega)|} \right| < \epsilon \quad \wedge \quad \max_{\omega \in \Omega} \left| \frac{x^{k+1}(\omega) - x^k(\omega)}{1 + |x^{k+1}(\omega)|} \right| < \epsilon$$

   then stop; else increment the iteration counter k by one and go to step 2.

Markov-Perfect Industry Dynamics

- Ericson, R. & Pakes, A. (1995) "Markov-Perfect Industry Dynamics: A Framework for Empirical Work."
- EP model tracks evolution of oligopolistic industries.
- Special case of dynamic game:
  - Entry, exit, and investment decisions.
  - Product market competition.
- Captures key findings of empirical literature on industry evolution:
  - Entry and exit occur simultaneously.
  - Heterogeneity among firms evolves endogenously and persists.

Applications in IO and Other Fields

- Advertising (Doraszelski & Markovich 2007).
- Capacity accumulation (Besanko & Doraszelski 2004, Chen 2009, Ryan 2012, Besanko, Doraszelski, Lu & Satterthwaite 2010a, 2010b, Wilson 2012).
- Collusion (Fershtman & Pakes 2000, 2005, de Roos 2004).
- Competitive convergence (Langohr 2003).
- Consumer learning (Ching 2010).
- Corporate reputation (Abito, Besanko & Diermeier 2012).
- Learning by doing (Benkard 2004, Besanko, Doraszelski, Kryukov & Satterthwaite 2010, Besanko, Doraszelski & Kryukov 2013).
- Mergers (Berry & Pakes 1993, Gowrisankaran 1999, Mermelstein, Nocke, Satterthwaite & Whinston 2013).
- Network effects (Jenkins, Liu, Matzkin & McFadden 2004, Markovich 2004, Markovich & Moenius 2005, Chen, Doraszelski & Harrington 2009).
- Productivity growth (Laincz 2005).
- R&D (Gowrisankaran & Town 1997, Auerswald 2001, Song 2011).
- Switching costs (Chen 2011).
- Technology adoption (Schivardi & Schneider 2005).
- International trade (Erdem & Tybout 2003).
- Finance (Goettler, Parlour & Rajan 2004).

Connections to Operations Research and Applied Math Literatures

- Discrete-time games go back to Shapley (1953), continuous-time games to Isaacs (1954).

- Markov perfect equilibrium (Maskin & Tirole 2001) "rediscovers" feedback Nash equilibrium.

- Lots of existence proofs (Sobel 1971, Federgruen 1976, Whitt 1980).

- Less on algorithms.

- Not everything is useful for economics (zero-sum games, average-payoff games).

- Good textbooks: Filar & Vrieze (1997), Basar & Olsder (1999).

Connections to Economics Literature

- EP model combines literature on long-run industry equilibrium (Jovanovic 1982, Hopenhayn 1992, Melitz 2003) with game theory (Tirole 1988, Fudenberg & Tirole 1991).

- EP model builds on analytically tractable special cases of dynamic games:

  - exponential games (Loury 1979, Lee & Wilde 1980, Reinganum 1982).

  - linear-quadratic games (Friedman 1983, Fershtman 1984, Reynolds 1987, 1991, Dockner 1992).

Application to Capacity Accumulation

- Besanko, D. & Doraszelski, U. (2004) "Capacity Dynamics and Endogenous Asymmetries in Firm Size."
- Substantial and persistent differences in firm sizes despite idiosyncratic shocks (Gort 1963, Mueller 1986, McGahan & Porter 1997).
- Size differences can arise endogenously in asymmetric equilibria of two- or three-stage models of capacity choice (Saloner 1987, Maggi 1996, Reynolds & Wilson 2000).
- But: What happens if firms are subject to idiosyncratic shocks? What about feedback effects?
- Dynamic models of capacity accumulation:
  - Steady-state analysis (Spence 1979, Fudenberg & Tirole 1983).
  - Linear-quadratic games (Hanig 1985, Reynolds 1987, 1991, Dockner 1992).

Relationship to Quality Ladder Model

- State variables $\omega = (\omega_1, \omega_2)$ are capacities of firms 1 and 2.

- Firms invest in capacity. Capacity may depreciate.

- Product market competition:
  - Quantity competition subject to capacity constraints.
  - Price competition subject to capacity constraints.

Substantial and Persistent Differences in Firm Sizes

	quantity competition	**price competition**
irreversible investment ($\delta = 0$)	symmetric firms	slightly asymmetric firms
reversible investment ($\delta > 0$)	symmetric firms	hugely asymmetric firms

Investment Reversibility and Preemption Races

- "An open issue (...) is the behavior of investment in the industry when capital depreciates. Intuition suggests that capital ought to lose some of its commitment value and that the steady-state levels of capital should be less sensitive to the initial head start of one of the firms." (Tirole 1988, p. 345)

- This paper: Investment reversibility may make preemption races *more* attractive.

Investment Reversibility and Preemption Races

- What is the main difference between the two modes of product market competition?
  - Capacity-constrained quantity competition: a firm's profit *plateaus* in own capacity.
  - Capacity-constrained price competition: a firm's profit *peaks* in own capacity (provided rival has sufficient capacity).
- Under price competition, it is in the self-interest of a not-too-small firm to withdraw from the race once its rival has gained a size advantage over it.
- By building up its capacity, a firm hopes to gain an initial edge over its rival and to decide the race in its favor.
- A firm anticipates that once it gains an edge over its rival, its rival will withdraw capacity.
- It is easier to withdraw capacity if the rate of depreciation is high, and it is impossible to do so if the rate of depreciation is zero.

Application to Advertising Dynamics

- Doraszelski, U. & Markovich, S. (2007) "Advertising Dynamics and Competitive Advantage."

- Can advertising lead to a sustainable competitive advantage?

- Existing *static* models of advertising competition (Butters 1977, Grossman & Shapiro 1984, Boyer & Moreaux 1999) cannot address this question.

- Existing *dynamic* models of advertising competition (Friedman 1983, Fershtman 1984, Cellini & Lambertini 2003) say no (globally stable symmetric steady state).

- This paper: Yes!

Goodwill and Awareness Advertising

- Consumer m's problem is to choose among the products in his choice set C_m such that

$$\max_{n \in C_m} \left(v_n - p_n + \epsilon_{mn} \right).$$

- Goodwill advertising influences the utility that consumers derive from the product.
  - Persuasive advertising (Dixit & Norman 1978).
  - Complementary advertising (Stigler & Becker 1977, Becker & Murphy 1993).

- Awareness advertising influences the share of consumers who are aware of the product.
  - Informative advertising (Stigler 1961, Butters 1977, Grossman & Shapiro 1984).

Relationship to Quality Ladder Model

- Goodwill advertising: State variables $\mathbf{v} = (v_1, v_2)$ are perceived qualities of firms 1 and 2.
- Awareness advertising: State variables $\mathbf{s} = (s_1, s_2)$ are shares of consumers who are aware of firms 1 and 2.
- Firms invest in advertising. Goodwill/awareness may depreciate.
- Product market competition: Price competition with differentiated products.

Sustainable Competitive Advantage: Goodwill Advertising

small market/ expensive advertising	**large market/ cheap advertising**
extremely asymmetric firms	symmetric firms

Goodwill Advertising and Cost/Benefit Considerations

- Marginal benefit of advertising is determined by

$$\pi(v_1 + \Delta, v_2) - \pi(v_1, v_2)$$

  and is proportional to market size.

- In a small market, the marginal benefit is small.

- Marginal benefit is decreasing in rival's goodwill $\rightarrow$ large firm can deter small firm.

- Marginal benefit is increasing in firm's goodwill $\rightarrow$ large firm cannot deter medium or large firm.

Sustainable Competitive Advantage: Awareness Advertising

low perceived quality	**high perceived quality**
symmetric firms	asymmetric firms

Awareness Advertising and Product Market Competition

- What is the main difference between low and high perceived quality?
  - Low perceived quality: a firm's profit *increases* in own awareness.
  - High perceived quality: a firm's profit first increases then *decreases* in own awareness (provided rival has sufficient awareness).

- Strategic advantage is grounded in product market competition.
  - Perceived quality of firms' products and intensity of competition.
  - Captive segment vs competitive segment: The probability of buying from firm 1 is

$$D_1(p_1, p_2; s_1, s_2)$$
$$= s_1(1 - s_2) \underbrace{\frac{\exp(v - p_1)}{1 + \exp(v - p_1)}}_{\text{captive segment}} + s_1 s_2 \underbrace{\frac{\exp(v - p_1)}{1 + \exp(v - p_1) + \exp(v - p_2)}}_{\text{competitive segment}}.$$

  - If $s_2 = 0$ ($s_2 = 1$), firm 1 set its monopolistic (duopolistic) price.
  - More generally, $s_2 \uparrow \rightarrow p_1^* \downarrow \rightarrow p_2^* \downarrow$.

Awareness Advertising and Product Market Competition

- With high perceived quality, a medium firm is better off staying put $\rightarrow$ large firm can deter medium firm.

- With high perceived quality, a small firm is better off trying to grow $\rightarrow$ large firm cannot deter small firm.

- Strategic advantage is independent of cost/benefit considerations.

Existence, Purification, and Multiplicity of Equilibrium

- Doraszelski, U. & Satterthwaite, M. (2010) "Computable Markov-Perfect Industry Dynamics."

- Questions:
  - Does a MPE exist in the EP model?
  - Is the MPE computationally tractable?
    - Pure strategies.
    - Symmetric and anonymous (exchangeable).
  - Is the MPE unique?

- Answers:
  - In the EP model a symmetric and anonymous MPE in pure strategies always exists under reasonable conditions.
  - The MPE is not necessarily unique.

Three Difficulties

- Randomization over discrete actions (entry/exit):
  - Introduce randomly drawn, privately-known setup costs/scrap values $\rightarrow$ the game of incomplete information has a MPE in cutoff entry/exit strategies.
- Randomization over continuous actions (investment):
  - Provide conditions on the model's primitives (UIC admissibility) such that a firm's optimal investment level is always unique $\rightarrow$ the MPE is in pure investment strategies.
  - Recent generalization: Escobar, J. (2013) "Equilibrium Analysis of Dynamic Models of Imperfect Competition."
- Symmetry and anonymity.
  - Provide conditions on the model's primitives $\rightarrow$ the MPE is symmetric and anonymous.

Quality Ladder Model with Entry/Exit

- Pakes, A. & McGuire, P. (1994) "Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model."

- Borkovsky, R., Doraszelski, U. & Kryukov, Y. (2012) "A Dynamic Quality Ladder Model with Entry and Exit: Exploring the Equilibrium Correspondence Using the Homotopy Method."

- Incumbent firms (i.e., active firms) and potential entrants (i.e., inactive firms).

- Two firms that can be either a potential entrant or an incumbent firm with potentially different product qualities

$$\omega = (\omega_1, \omega_2) \in \{ \underbrace{1, \ldots, L}_{\text{active firm}}, \underbrace{L+1}_{\text{inactive firm}} \}^2 = \Omega.$$

- Exit is a transition from state $\omega_n \neq L+1$ to state $\omega'_n = L+1$.

- Entry is a transition from state $\omega_n = L+1$ to state $\omega'_n = \omega^e \neq L+1$, where ω^e is an exogenously given initial product quality.

Quality Ladder Model with Entry/Exit

- Let $\xi_n(\omega) \in [0,1]$ be firm n's probability of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry.

- Transition probability: If $\omega_n \in \{2,\ldots,L-1\}$, then

$$\Pr(\omega_n'|\omega_n,\xi_n,x_n) = \begin{cases} \xi_n \frac{(1-\delta)\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n + 1, \\ \xi_n \frac{1-\delta+\delta\alpha x_n}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n, \\ \xi_n \frac{\delta}{1+\alpha x_n} & \text{if} \quad \omega_n' = \omega_n - 1, \\ 1-\xi_n & \text{if} \quad \omega_n' = L+1, \end{cases}$$

etc. If $\omega_n = L+1$, then

$$\Pr(\omega_n'|\omega_n,\xi_n) = \begin{cases} \xi_n & \text{if} \quad \omega_n' = \omega^e, \\ 1-\xi_n & \text{if} \quad \omega_n' = L+1. \end{cases}$$

Quality Ladder Model with Entry/Exit

- Firm n is assigned a random scrap value $\phi_n \sim F$ (if $\omega_n \neq L+1$) or a random setup cost $\phi_n^e \sim F^e$ (if $\omega_n = L+1$).
  - Scrap values/setup costs are private information.
  - Scrap values/setup costs are independent across firms and periods.
- Because scrap values and setup costs are private to a firm, its rivals perceive the firm *as if* it is mixing.
- In each period the timing is as follows:
  - Incumbent firms learn their scrap value and decide on exit and investment. Potential entrants learn their setup cost and decide on entry and investment.
  - Incumbent firms compete in the product market.
  - Exit and entry decisions are implemented.
  - The investment decisions of the remaining incumbents and new entrants are carried out and their uncertain outcomes are realized.

Incumbent Firm

- Bellman equation without entry/exit:

$$V_n(\omega) = \max_{x_n \geq 0} \pi_n(\omega) - x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n).$$

- Bellman equation with entry/exit:

$$V_n(\omega) = \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - \xi_n) \mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\}$$
$$+\xi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\},$$

  where

$$(1 - \xi_n) \mathrm{E}\left\{\phi_n | \phi_n \geq F^{-1}(\xi_n)\right\} = \int_{\phi_n \geq F^{-1}(\xi_n)} \phi_n dF(\phi_n).$$

- An optimizing incumbent cares about the scrap value conditional on receiving it.

- Optimality condition:

$$\xi_n(\omega) = F\left(-x_n(\omega) + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \mathrm{Pr}(\omega_n' | \omega_n, x_n(\omega), \xi_n = 1) \right).$$

Incumbent Firm: Derivation of Bellman Equation

- Let $\chi_n(\omega, \phi_n) \in \{0, 1\}$ be firm n's decision of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry.

- Let $\xi_n(\omega) = \int \chi_n(\omega, \phi_n) dF(\phi_n)$ be firm n's probability of remaining in (if $\omega_n \neq L+1$) or entering into (if $\omega_n = L+1$) the industry (as perceived by its rivals).

- Let $V_n(\omega, \phi_n)$ be the value function of incumbent firm n *after* it observes its scrap value.

- Bellman equation:

$$V_n(\omega, \phi_n) = \max_{\chi_n \in \{0,1\}, x_n \geq 0} \pi_n(\omega) + (1 - \chi_n)\phi_n$$

$$+\chi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\},$$

- The problem of the incumbent can be broken up into two parts:
  - The incumbent chooses its investment conditional on remaining in the industry $\rightarrow$ the optimal investment choice is independent of the firm's scrap value.
  - Given its investment choice, the incumbent decides whether or not to remain in the industry.

Incumbent Firm: Derivation of Bellman Equation

- Optimal decision has reservation property:

$$\chi_n(\omega, \phi_n) = \begin{cases} 1 & \text{if} \quad \phi_n \leq \bar{\phi}_n(\omega), \\ 0 & \text{if} \quad \phi_n \geq \bar{\phi}_n(\omega), \end{cases}$$

where

$$\bar{\phi}_n(\omega) = \max_{x_n \geq 0} -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1)$$

denotes the cutoff scrap value.

- Restrict attention to decision rules of the form $1[\phi_n < \bar{\phi}_n(\omega)]$.

- Instead of the cutoff $\bar{\phi}_n(\omega)$, represent these rules with the induced probability $\xi_n(\omega)$:

$$\xi_n(\omega) = \int \chi(\omega, \phi_n) dF(\phi_n) = \int 1[\phi_n < \bar{\phi}_n(\omega)] dF(\phi_n) = F(\bar{\phi}_n(\omega))$$
$$\Leftrightarrow \bar{\phi}_n(\omega) = F^{-1}(\xi_n(\omega))$$

provided F has positive density on its support.

Incumbent Firm: Derivation of Bellman Equation

- Imposing the reservation property on the Bellman equation yields

$$V_n(\omega, \phi_n) = \max_{\chi_n \in \{0,1\}, x_n \geq 0} \pi_n(\omega) + (1 - \chi_n)\phi_n$$

$$+\chi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\}$$

$$= \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - 1[\phi_n < F^{-1}(\xi_n)])\phi_n$$

$$+1[\phi_n < F^{-1}(\xi_n)] \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\}.$$

Incumbent Firm: Derivation of Bellman Equation

- Let $V_n(\omega) = \int V_n(\omega, \phi_n) dF(\phi_n)$ be the value function of incumbent firm n *before* it observes its scrap value.

- Integrating over ϕ_n on both sides of the Bellman equation yields

$$V_n(\omega) = \int \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + (1 - 1[\phi_n < F^{-1}(\xi_n)])\phi_n$$

$$+1[\phi_n < F^{-1}(\xi_n)] \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\} dF(\phi_n)$$

$$= \max_{\xi_n \in [0,1], x_n \geq 0} \pi_n(\omega) + \int_{\phi_n \geq F^{-1}(\xi_n)} \phi_n dF(\phi_n)$$

$$+\xi_n \left\{ -x_n + \beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega)) \Pr(\omega_n' | \omega_n, x_n, \xi_n = 1) \right\}.$$

Potential Entrant

- Potential entrants are short-lived.

- Bellman equation:

$$V_n(\omega) = \max_{\xi_n \in [0,1]} \xi_n \Bigg\{ -\mathsf{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\}$$
$$+\beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega))\mathsf{Pr}(\omega_n'|\omega_n, \xi_n = 1)\Bigg\},$$

where

$$\xi_n \mathsf{E}\left\{\phi_n^e | \phi_n^e \leq F^{e-1}(\xi_n)\right\} = \int_{\phi_n^e \leq F^{e-1}(\xi_n)} \phi_n^e dF^e(\phi_n^e).$$

- An optimizing entrant cares about the setup cost conditional on paying it.

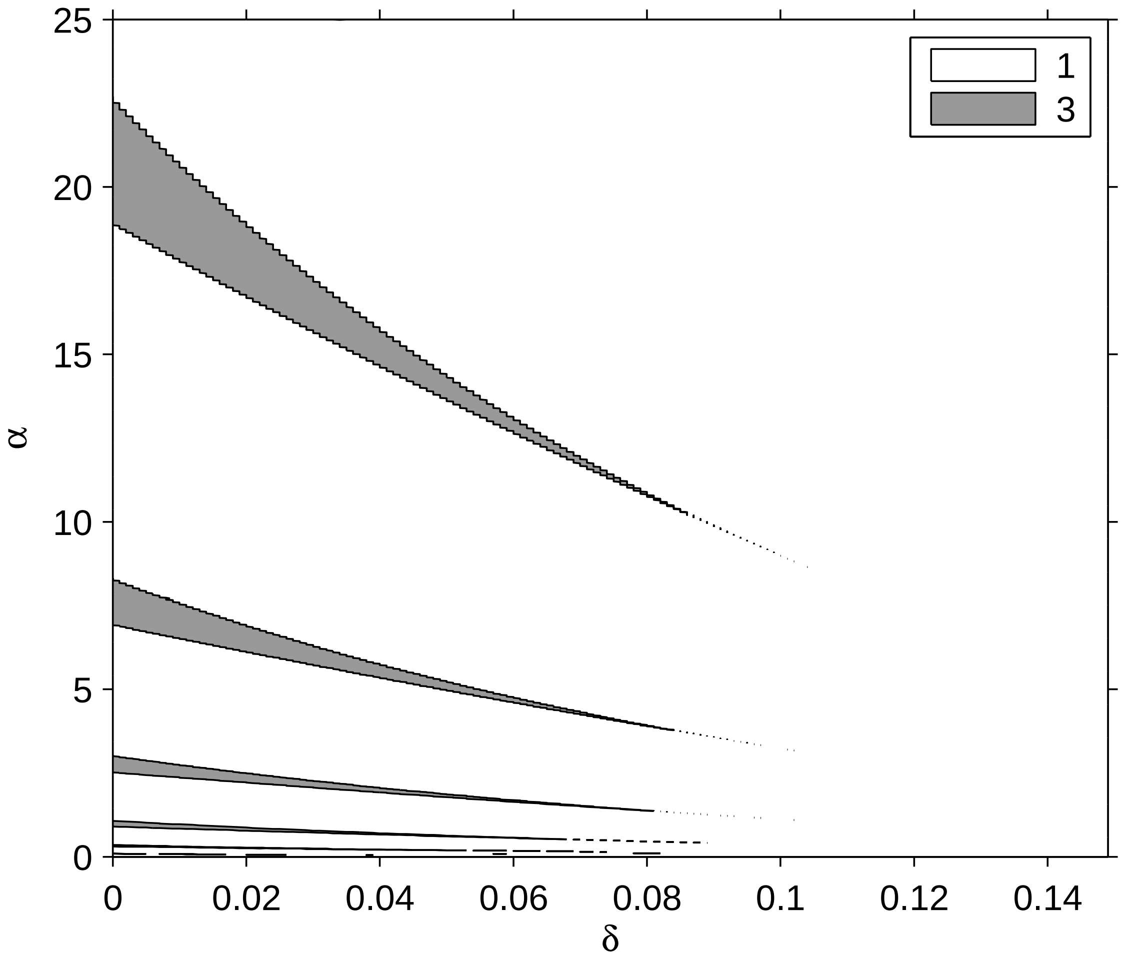
- Optimality condition:

$$\xi_n(\omega) = F^e\left(\beta \sum_{\omega_n'=1}^{L} W_n(\omega_n'; \omega_{-n}, \xi_{-n}(\omega), x_{-n}(\omega))\mathsf{Pr}(\omega_n'|\omega_n, \xi_n = 1)\right).$$

Multiple Equilibria

. . . we have experimented quite a bit with the core version of the algorithm, and we never found two sets of equilibrium policies for a given set of primitives (we frequently run the algorithm several times using different initial conditions or different orderings of points looking for other equilibria that might exist). We should emphasize here that the core version, and indeed most other versions that have been used, all use quite simple functional forms for the primitives of the problem, and multiplicity of equilibrium may well be more likely when more complicated functional forms are used. Of course, most applied work suffices with quite simple functional forms. (Pakes 2000, pp. 18–19)

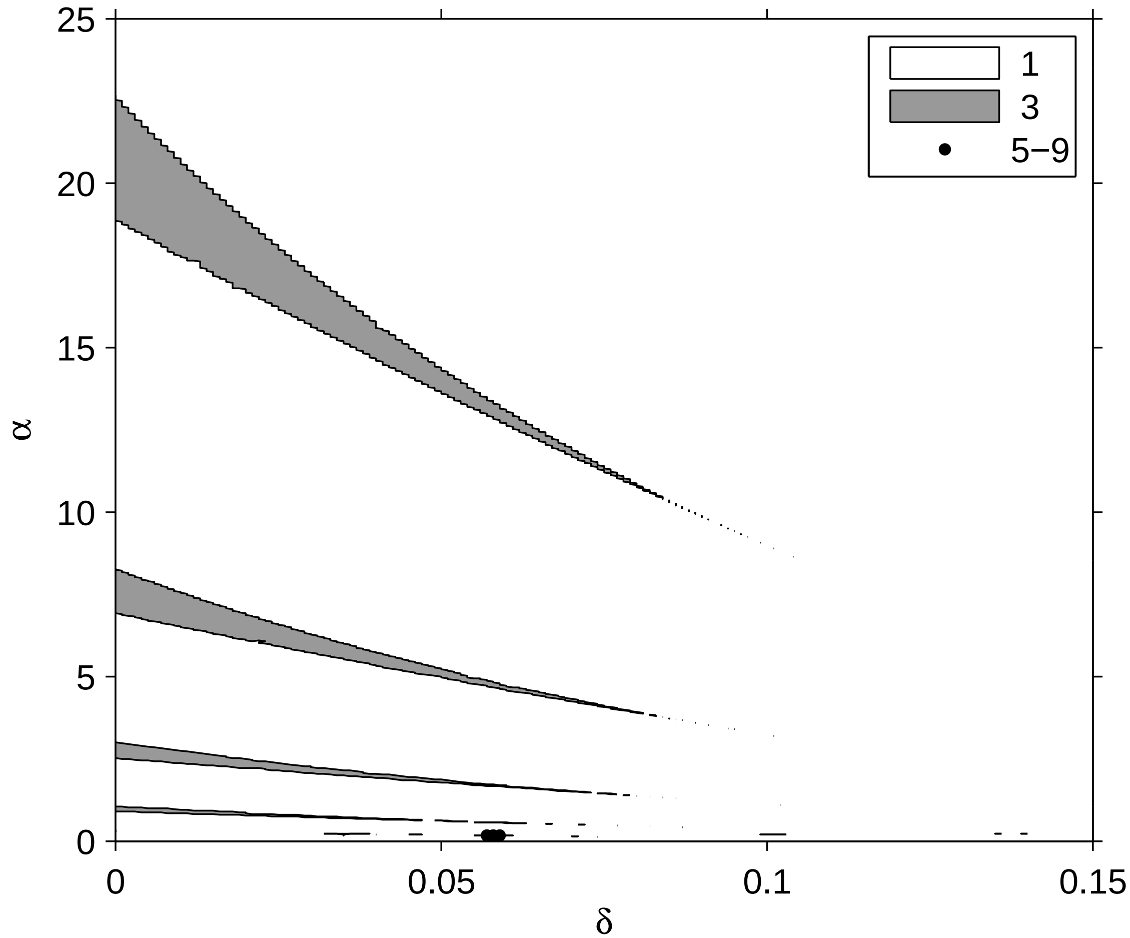
Multiple Equilibria: Quality Ladder Model without Entry/Exit



Number of equilibria in the Pakes & McGuire (1994) quality ladder model without entry/exit.

Source: Borkovsky, Doraszelski & Kryukov (2010).

Multiple Equilibria: Quality Ladder Model with Entry/Exit



Number of equilibria in the Pakes & McGuire (1994) quality ladder model with entry/exit. Source: Borkovsky, Doraszelski & Kryukov (2012).

Multiple Equilibria

...I should note that virtually all Markov Perfect Models have multiple equilibria...(anonymous referee, 2013)