# Dynamic Games: Numerical Methods and Applications

Day 2a

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## Agenda

- Application: Learning-by-doing.
- Computing all equilibria: Homotopy method.

## Multiple Equilibria

. . . I should note that virtually all Markov Perfect Models have multiple equilibria. . . (anonymous referee, 2013)

## Multiple Equilibria in Estimation

Nested-fixed point algorithm:

- Gowrisankaran, G. & Town, R. (1997) "Dynamic equilibrium in the hospital industry."

Two-step methods:

- Aguirregabiria, V. & Mira, P. (2007) "Sequential Estimation of Dynamic Discrete Games."

  Additional readings:

  - Pesendorfer, M. & Schmidt-Dengler, P. (2010) "Sequential Estimation of Dynamic Discrete Games: A Comment."
  - Kasahara, H. & Shimotsu, K. (2012) "Sequential Estimation of Structural Models with a Fixed Point Constraint."

- Bajari, P., Benkard, L. & Levin, J. (2007) "Estimating Dynamic Models of Imperfect Competition."

- Pakes, A., Ostrovsky, M. & Berry, S. (2007) "Simple Estimators for the Parameters of Discrete Dynamic Games (with Entry/Exit Examples)."

- Pesendorfer, M. & Schmidt-Dengler, P. (2008) "Asymptotic Least Squares Estimators for Dynamic Game."

MPEC method:

- Judd, K. & Su, C. (2012) "Constrained Optimization Approaches to Estimation of Structural Models."

## Multiple Equilibria in Counterfactual Analysis

Out-of-equilibrium adjustment processes:

- Lee, R. & Pakes, A. (2009) "Multiple Equilibria and Selection by Learning in an Applied Setting."

- Doraszelski, U. & Escobar, J. (2010) "A Theory of Regular Markov Perfect Equilibria in Dynamic Stochastic Games: Genericity, Stability, and Purification."

- Aguirregabiria, V. (2012) "A Method for Implementing Counterfactual Experiments in Models with Multiple Equilibria."

## Learning-by-Doing

- Besanko, D., Doraszelski, U., Kryukov, S. & Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."

- Question: Is organizational forgetting an antidote to market dominance?

- Incorporate organizational forgetting into the Cabral & Riordan (1994) model of learning-by-doing.

- Dynamic competition with learning-by-doing and organizational forgetting is akin to racing down an upward-moving escalator.

- Organizational forgetting makes bidirectional movements through the state space possible. Thus, it is a source of...
  - ...aggressive pricing behavior;
  - ...market dominance;
  - ...multiple equilibria.

- Learning-by-doing and organizational forgetting are distinct economic forces.

## Learning-by-Doing

- Discrete time, infinite horizon.
- Two firms with potentially different stocks of know-how

$$\omega = (\omega_1, \omega_2) \in \{1, \ldots, L\}^2 = \Omega.$$

- In each period, the timing is as follows:
  - Firms choose prices.
  - One buyer enters the market and makes at most one purchase.
  - Learning-by-doing and organizational forgetting occur and the firms' stocks of know-how change accordingly.
- Law of motion:

$$\omega_n' = \omega_n + q_n - f_n,$$

  where
  - $q_n \in \{0, 1\}$ indicates whether firm $n$ makes a sale with

$$\Pr(q_n = 1) = D_n(p_1, p_2) = \frac{\exp(v - p_n)}{1 + \sum_{k=1}^{2} \exp(v - p_k)};$$

  - $f_n \in \{0, 1\}$ represents organizational forgetting with

$$\Pr(f_n = 1) = \Delta(\omega_n) = 1 - (1 - \delta)^{\omega_n}.$$

## Bellman Equation

- Let $V_n(\omega)$ denote the expected NPV to firm $n$ if the current state is $\omega$.
- Firm $n$'s Bellman equation is

$$V_n(\omega) = \max_{p_n} D_n(p_n, p_{-n}(\omega))(p_n - c(\omega_n)) + \beta \sum_{k=0}^{2} D_k(p_n, p_{-n}(\omega)) W_{nk}(\omega),$$

  where

  - $p_{-n}(\omega)$ is the price charged by the other firm;
  - the marginal cost of production is

$$c(\omega_n) = \begin{cases} \kappa \omega_n^{\eta} & \text{if} \quad 1 \leq \omega_n < l, \\ \kappa l^{\eta} & \text{if} \quad l \leq \omega_n \leq L, \end{cases}$$

    with $\eta = \log_2 \rho$ for a progress ratio of $\rho$;
  - $\beta \in (0, 1)$ is the discount factor;
  - $W_{nk}(\omega)$ is the expectation of firm $n$'s value function conditional on buyer purchasing good $k \in \{0, 1, 2\}$ (good 0 is outside good).

## Bellman Equation

- Continuation values:

$$W_{n0}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \text{Pr}(\omega_1'|\omega_1, q_1 = 0) \text{Pr}(\omega_2'|\omega_2, q_2 = 0),$$

$$W_{n1}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \text{Pr}(\omega_1'|\omega_1, q_1 = 1) \text{Pr}(\omega_2'|\omega_2, q_2 = 0),$$

$$W_{n2}(\omega) = \sum_{\omega_1'=1}^{L} \sum_{\omega_2'=1}^{L} V_n(\omega') \text{Pr}(\omega_1'|\omega_1, q_1 = 0) \text{Pr}(\omega_2'|\omega_2, q_2 = 1),$$

where

$$\text{Pr}(\omega_n'|\omega_n, q_n) = \begin{cases} 1 - \Delta(\omega_n) & \text{if} \quad \omega_n' = \omega_n + q_n, \\ \Delta(\omega_n) & \text{if} \quad \omega_n' = \omega_n + q_n - 1, \end{cases}$$

and $\text{Pr}(L|L, q_n = 1) = 1$ and $\text{Pr}(1|1, q_n = 0) = 1$.

## Pricing Strategy

- $p_n(\omega)$ is unique solution to FOC:

$$0 = 1 - (1 - D_n(p_n, p_{-n}(\omega)))\,(p_n - c(\omega_n)) - \beta W_{nn}(\omega) + \beta \sum_{k=0}^{2} D_k(p_n, p_{-n}(\omega)) W_{nk}(\omega).$$

- No closed-firm solution. Solve numerically.

## Equilibrium

- Primitives are symmetric.

- Symmetric Markov perfect equilibrium (MPE):
  - Value function $V_1(\omega_1, \omega_2) = V(\omega_1, \omega_2)$ and $V_2(\omega_1, \omega_2) = V(\omega_2, \omega_1)$.
  - Policy function $p_1(\omega_1, \omega_2) = p(\omega_1, \omega_2)$ and $p_2(\omega_1, \omega_2) = p(\omega_2, \omega_1)$.

- Existence in pure strategies is guaranteed (Doraszelski & Satterthwaite 2010), uniqueness is not.

- The goal is to compute the value and policy functions (or, more precisely, $L \times L$ matrices) $\mathbf{V}$ and $\mathbf{p}$.

# Computation: Pakes & McGuire (1994) Algorithm

1. Make initial guesses $\mathbf{V}^0$ and $\mathbf{p}^0$, choose a dampening factor $\lambda \in (0,1]$, choose a stopping criterion $\epsilon > 0$, and initialize the iteration counter to $l = 1$.

2. For all states $\omega \in \Omega$ compute

$$\begin{aligned} p^{l+1}(\omega) &= \arg\max_{p_1} D_1(p_1, p^l(\omega_2,\omega_1))\,(p_1 - c(\omega_1)) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p_1, p^l(\omega_2,\omega_1)) W_k^l(\omega) \end{aligned}$$

   and

$$\begin{aligned} V^{l+1}(\omega) &= D_1(p^{l+1}(\omega), p^l(\omega_2,\omega_1))\left(p^{l+1}(\omega) - c(\omega_1)\right) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p^{l+1}(\omega), p^l(\omega_2,\omega_1)) W_k^l(\omega). \end{aligned}$$

3. Dampening step: Assign

$$\begin{aligned} \mathbf{V}^{l+1} &\leftarrow \lambda \mathbf{V}^{l+1} + (1-\lambda)\mathbf{V}^l, \\ \mathbf{p}^{l+1} &\leftarrow \lambda \mathbf{p}^{l+1} + (1-\lambda)\mathbf{p}^l. \end{aligned}$$

4. If

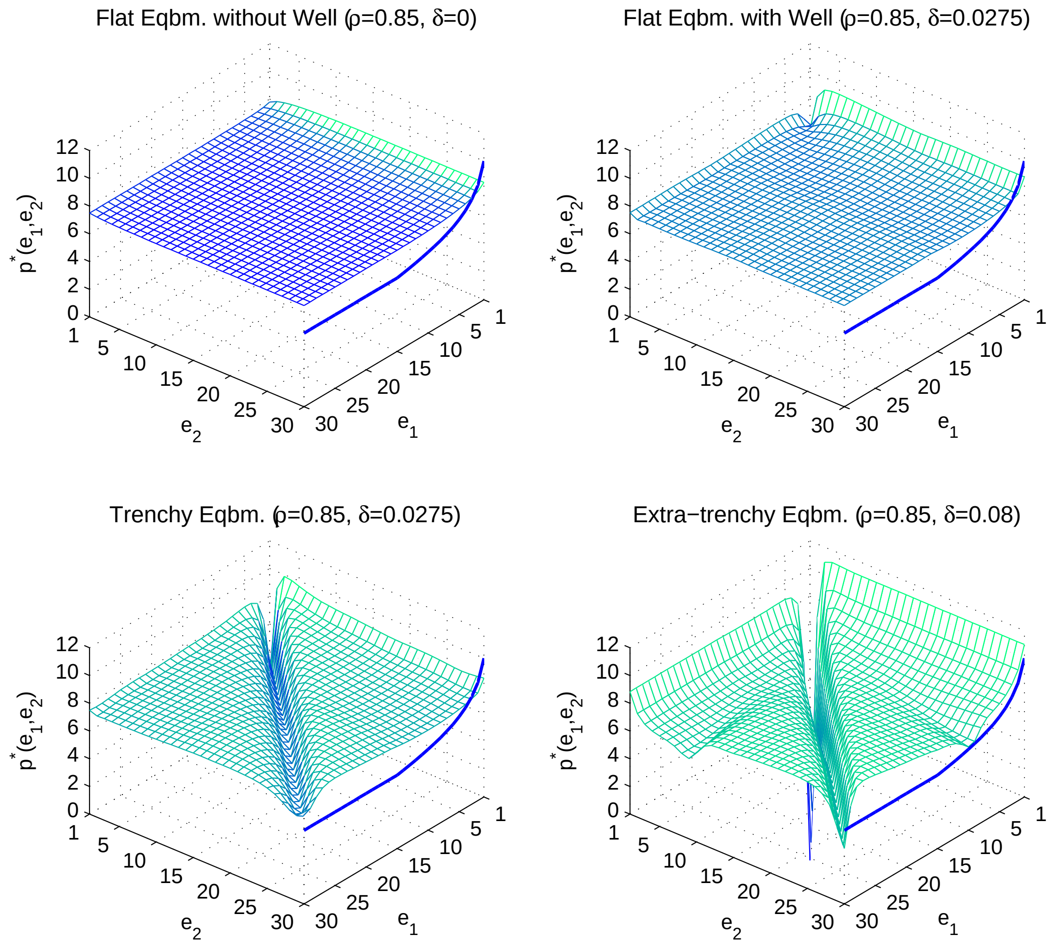
$$\max_{\omega\in\Omega} \left| \frac{V^{l+1}(\omega) - V^l(\omega)}{1 + |V^{l+1}(\omega)|} \right| < \epsilon \quad \wedge \quad \max_{\omega\in\Omega} \left| \frac{p^{l+1}(\omega) - p^l(\omega)}{1 + |p^{l+1}(\omega)|} \right| < \epsilon$$

   then stop; else increment the iteration counter $l$ by one and go to step 2.

# Categories of Equilibria

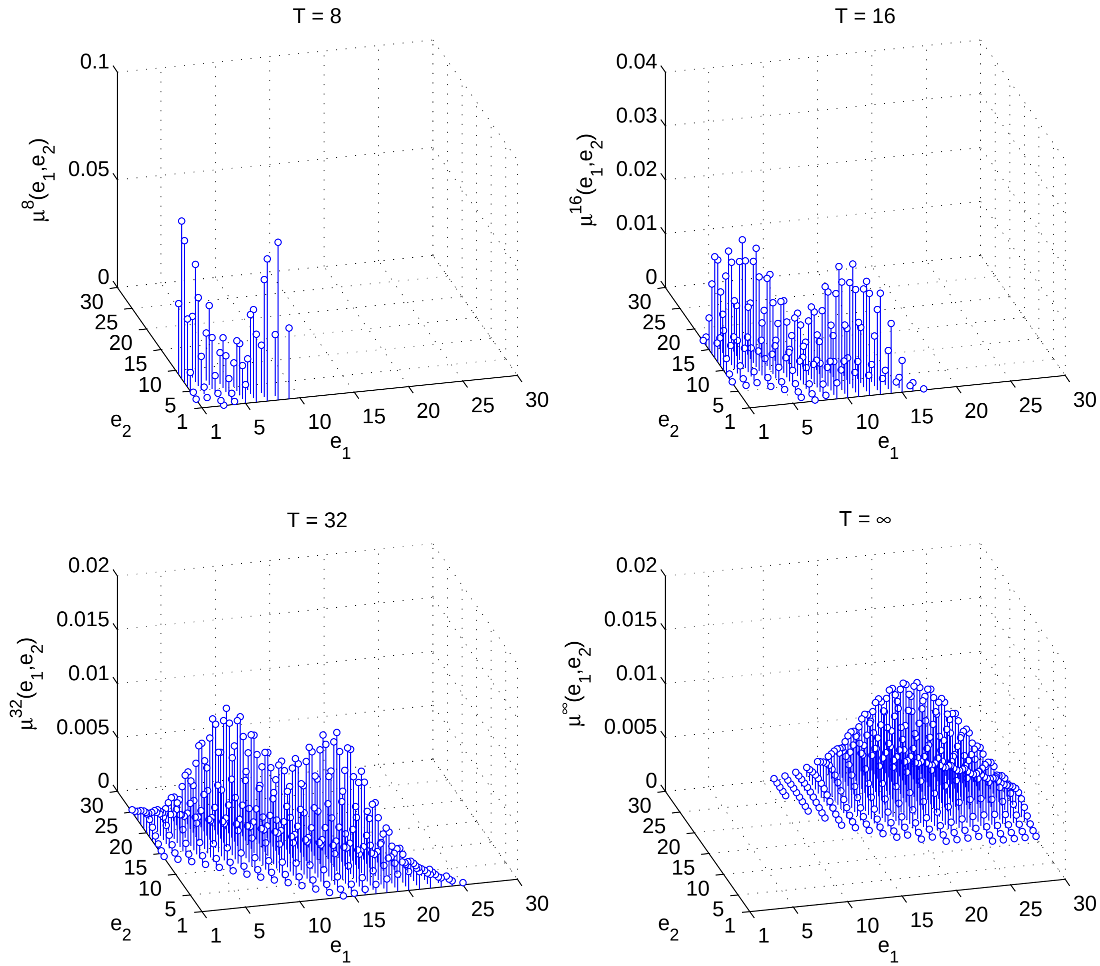
Categories of equilibria:

- flat without well;
- flat with well;
- trenchy;
- extra-trenchy.



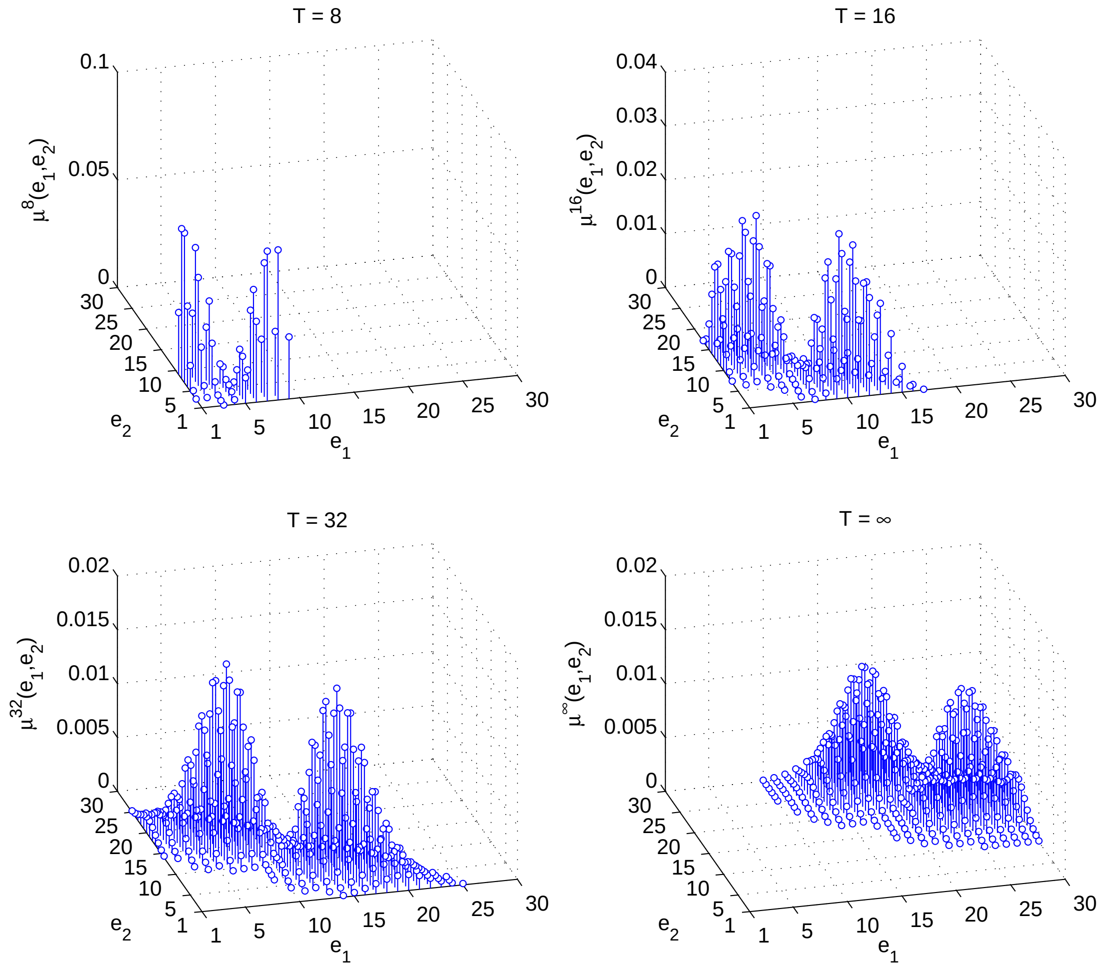

Policy function $p^*(\omega_1, \omega_2)$. Marginal cost $c(\omega_1)$ (solid line in $\omega_2 = 30$-plane).

# Industry Dynamics: Flat Equilibrium with Well ($\rho = 0.85$, $\delta = 0.03$)




Transient distribution over states in periods 8, 16, and 32 given initial state $(1, 1)$ and limiting distribution.

# Industry Dynamics: Trenchy Equilibrium ($\rho = 0.85$, $\delta = 0.03$)

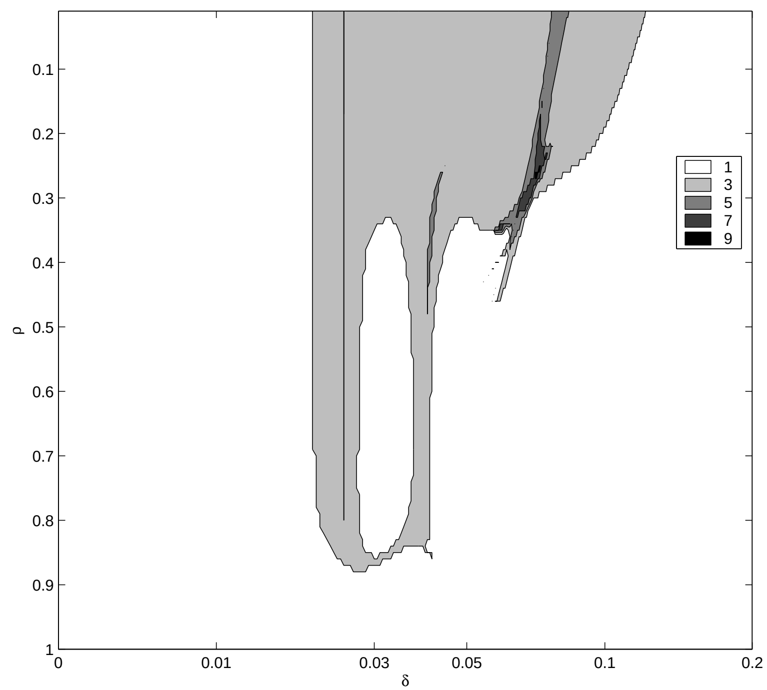



Transient distribution over states in periods 8, 16, and 32 given initial state $(1, 1)$ and limiting distribution.

## Multiple Equilibria

**Proposition 1** *If organizational forgetting is either absent ($\delta = 0$) or certain ($\delta = 1$), then there is a unique equilibrium.*

**Result 1** *If organizational forgetting is neither absent ($\delta = 0$) nor certain ($\delta = 1$), then there may be multiple equilibria.*




Number of equilibria.

## Organizational Forgetting and Multiple Equilibria

- What gives rise to multiple equilibria ranging from "peaceful coexistence" to "trench warfare"?

- Holding the value of continued play fixed, the strategic situation in state $\omega$ is akin to a static game.

  **Proposition 2** *Statewise uniqueness holds provided the outside good is sufficiently unattractive ($v$ large).*

- Multiple equilibria must arise from firms' expectations regarding the value of continued play.

  Taking the value of continued play as given, the reaction functions intersect once, but there is more than one value of continued play that is consistent with rational expectations.

- Multiplicity is rooted in the dynamics of the model.

## Organizational Forgetting and Multiple Equilibria

- When do multiple equilibria arise?

- In expectation, the "inflow" of know-how into the industry is almost one unit per period, the "outflow" in state $\omega$ is $\Delta(\omega_1) + \Delta(\omega_2)$.

- Consider state $(\omega, \omega)$, where $\omega \geq l$.
  - If $1 \ll 2\Delta(\omega)$, then it is virtually impossible that both firms reach the bottom of their learning curves $\rightarrow$ trench warfare.
  - If $1 \gg 2\Delta(\omega)$, then it is virtually inevitable that both firms reach the bottom of their learning curves $\rightarrow$ peaceful coexistence.
  - If $1 \approx 2\Delta(\omega)$, then primitives do not suffice to tie down the equilibrium $\rightarrow$ multiple equilibria.

    Back-of-the-envelope calculation ($l = 15$ and $L = 30$):

$$1 = 2\Delta(15) \Rightarrow \delta = 0.05 \text{ and } 1 = 2\Delta(30) \Rightarrow \delta = 0.02.$$

- Stagewise uniqueness and unidirectional movements through the state space $\rightarrow$ unique equilibrium.

  Organizational forgetting makes bidirectional movements possible.

## Homotopy Method

- Besanko, D., Doraszelski, U., Kryukov, S. & Satterthwaite, M. (2010) "Learning-by-Doing, Organizational Forgetting, and Industry Dynamics."

  Additional reading:

  - Borkovsky, R., Doraszelski, U., & Kryukov, Y. (2010) "A User's Guide to Solving Dynamic Stochastic Games Using the Homotopy Method."

- Show that there are equilibria that the Pakes & McGuire (1994) algorithm cannot compute.

- Propose a homotopy algorithm to trace out the equilibrium correspondence.

## Homotopy Method: Learning-by-Doing

- Bellman equation and FOC for state $\omega$ are

$$V(\omega) = D_1(\omega)\left(p(\omega) - c(\omega_1)\right) + \beta \sum_{k=0}^{2} D_k(\omega) W_k(\omega),$$

$$0 = 1 - (1 - D_1(\omega))\left(p(\omega) - c(\omega_1)\right) - \beta W_1(\omega) + \beta \sum_{k=0}^{2} D_k(\omega) W_k(\omega),$$

where $D_k(\omega) = D_k(p(\omega), p(\omega_2, \omega_1))$, $k \in \{0, 1, 2\}$.

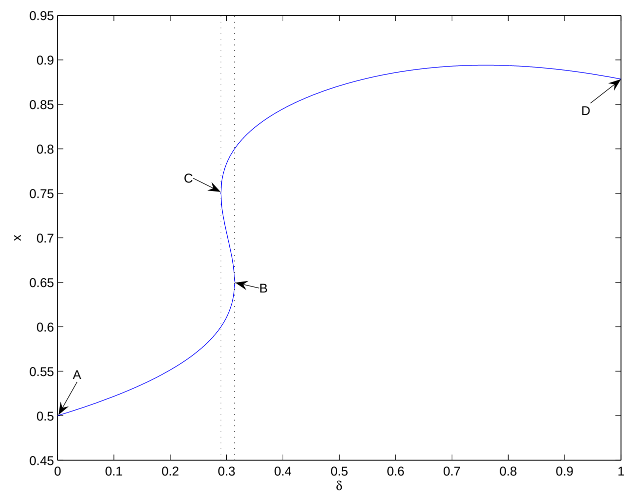
- The system of $2L^2$ nonlinear equations given by the collection of the above equations for each state $\omega \in \{1, \ldots, L\}^2$ defines a symmetric equilibrium.

# Homotopy Method: Learning-by-Doing

- Write the system of $2L^2$ nonlinear equations (Bellman equations and FOCs) as
$$\mathbf{F}(\mathbf{x},\delta) = 0,$$
where
$$\mathbf{x} = (V(1,1),\ldots,V(L,L),p(1,1),\ldots,p(L,L)).$$

- The object of interest is the equilibrium correspondence
$$\mathbf{F}^{-1} = \{(\mathbf{x},\delta)|\mathbf{F}(\mathbf{x},\delta) = 0\}.$$

- The homotopy algorithm follows a path from the unique equilibrium at $\delta = 0$ to the unique equilibrium at $\delta = 1$.




Equilibrium correspondence $\mathbf{F}^{-1}$ for simple example.

## Homotopy Method

- Define a parametric path to be a set of functions $(\mathbf{x}(s), \delta(s))$ such that $(\mathbf{x}(s), \delta(s)) \in \mathbf{F}^{-1}$.

- The conditions that are required to remain "on path" are found by differentiating
$$\mathbf{F}(\mathbf{x}(s), \delta(s)) = 0$$
with respect to $s$:
$$\sum_{i=1}^{2L^2} \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial x_i} x_i'(s) + \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \delta} \delta'(s) = 0.$$

- While there are many solutions, all of them describe the same path.

- One solution obeys the so-called basic differential equations (BDE)
$$y_i'(s) = (-1)^{i+1} \det \left( \left( \frac{\partial \mathbf{F}(\mathbf{y}(s))}{\partial \mathbf{y}} \right)_{-i} \right), \quad i = 1, \ldots, 2L^2 + 1, \tag{1}$$
where $\mathbf{y}(s) = (\mathbf{x}(s), \delta(s))$ and the notation $(\cdot)_{-i}$ is used to indicate that the $i$th column is removed from the $(2L^2 \times 2L^2 + 1)$ Jacobian $\frac{\partial \mathbf{F}(\mathbf{y}(s))}{\partial \mathbf{y}}$.

- The BDE reduce the task of tracing out the equilibrium correspondence to solving a system of differential equations.

## Homotopy Method: Simple Example

- Consider

$$F(x,\delta) = -15.289 - \frac{\delta}{1+\delta^4} + 67.500x - 96.923x^2 + 46.154x^3$$

with

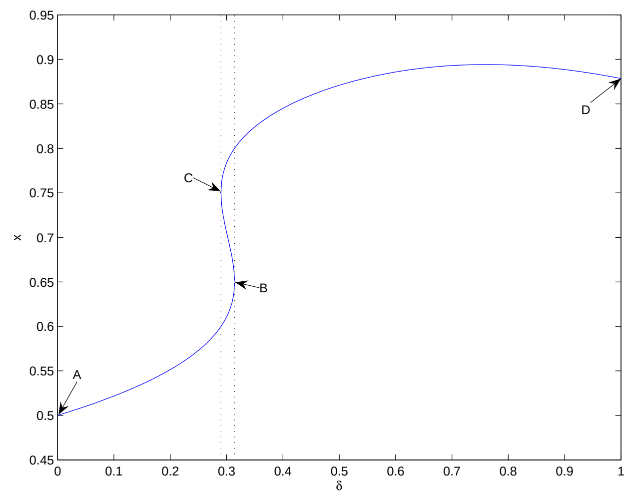
$$\frac{\partial F(x,\delta)}{\partial(x,\delta)} = \begin{pmatrix} 67.500 - 2\cdot 96.923x + 3\cdot 46.154x^2 & -\frac{1-3\delta^4}{(1+\delta^4)^2} \end{pmatrix}.$$

- Basic differential equations:

$$\begin{pmatrix} \frac{dx}{ds} \\ \frac{d\delta}{ds} \end{pmatrix} = \begin{pmatrix} \frac{\partial F(x,\delta)}{\partial \delta} \\ -\frac{\partial F(x,\delta)}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{1-3\delta^4}{(1+\delta^4)^2} \\ -67.500 + 2\cdot 96.923x - 3\cdot 46.154x^2 \end{pmatrix}$$

with initial condition $x(0) = 0.5$ and $\delta(0) = 0$.

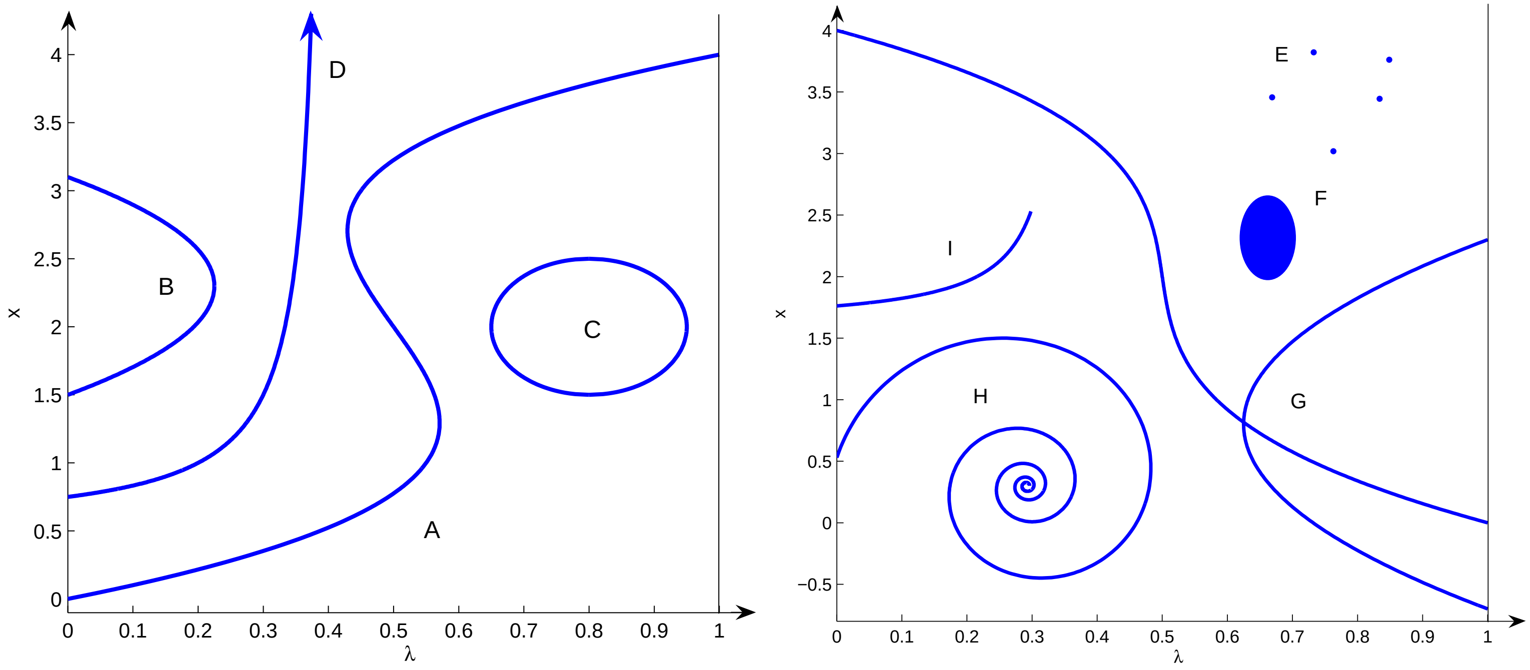
- Solve with e.g. finite-difference methods.




Equilibrium correspondence $\mathbf{F}^{-1}$ for simple example.

# Homotopy Method

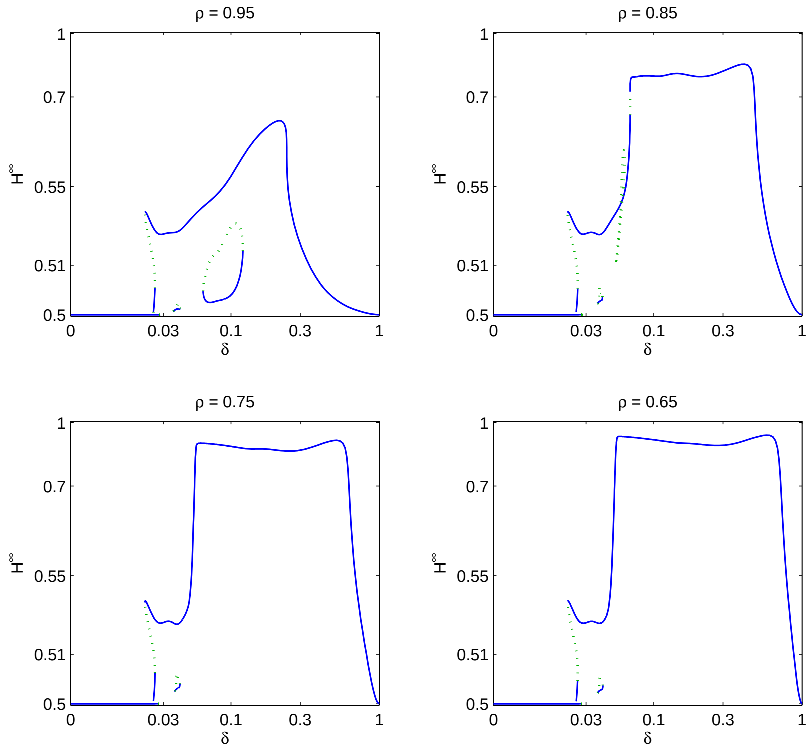
- The homotopy $\mathbf{F}$ is regular iff $\frac{\partial \mathbf{F}(\mathbf{y})}{\partial \mathbf{y}}$ has full rank at all points in $\mathbf{F}^{-1}$.




$\mathbf{F}^{-1}$ if $\mathbf{F}$ is regular (left panel) and irregular (right panel).

# Equilibrium Correspondence: Learning-by-Doing

**Result 2** *The equilibrium correspondence* $\mathbf{F}^{-1}$ *contains a unique path that connects the equilibrium at* $\delta = 0$ *with the equilibrium at* $\delta = 1$. *In addition,* $\mathbf{F}^{-1}$ *may contain (one or more) loops that are disjoint from this "main path" and from each other.*




Limiting expected Herfindahl index $H^\infty$. Equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) < 1$ (solid line) and equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) \geq 1$ (dotted line).

## Pakes & McGuire (1994) Algorithm

Executes the iteration

$$\mathbf{x}^{l+1} = \mathbf{G}(\mathbf{x}^l), \quad l = 0, 1, 2, \ldots,$$

where, for each state $\omega \in \{1, \ldots, L\}^2$, old guesses for the value and policy of firm 1 are mapped into new guesses as follows:

$$\begin{aligned} p^{l+1}(\omega) &= \arg\max_{p_1} D_1(p_1, p^l(\omega_2, \omega_1))\,(p_1 - c(\omega_1)) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p_1, p^l(\omega_2, \omega_1)) W_k^l(\omega), \\ V^{l+1}(\omega) &= D_1(p^{l+1}(\omega), p^l(\omega_2, \omega_1)) \left(p^{l+1}(\omega) - c(\omega_1)\right) \\ &\qquad + \beta \sum_{k=0}^{2} D_k(p^{l+1}(\omega), p^l(\omega_2, \omega_1)) W_k^l(\omega). \end{aligned}$$

## Pakes & McGuire (1994) Algorithm

- "Inbetween" two equilibria that can be computed using the Pakes & McGuire (1994) algorithm, there is one equilibrium that cannot:

**Proposition 3** *If* $\delta'(s) \leq 0$*, then* $\varrho\left(\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}\right) \geq 1$.

- Let $I$ denote the $(2L^2 \times 2L^2)$ identity matrix. Then

$$\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)} = \frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}} + I. \tag{2}$$

- The BDE (1) imply

$$\delta'(s) = \det\left(\frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}}\right).$$

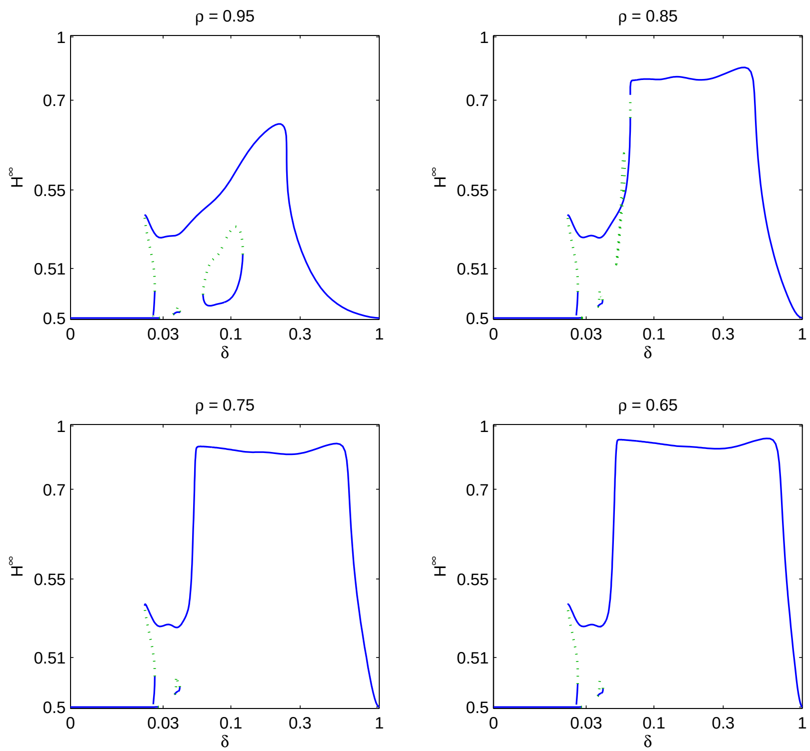
- Since the determinant of $\frac{\partial \mathbf{F}(\mathbf{x}(s), \delta(s))}{\partial \mathbf{x}}$ is the product of $2L^2$ eigenvalues, if $\delta'(s) \leq 0$, then there exists at least one real nonnegative eigenvalue.

- Let $A$ be an arbitrary matrix and $\varsigma(A)$ its spectrum. Then $\varsigma(A + I) = \varsigma(A) + 1$.

- It follows from equation (2) that $\left.\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\right|_{\delta(s)}$ has at least one real eigenvalue equal to or bigger than unity.

# Equilibrium Correspondence: Learning-by-Doing

**Result 3** *The equilibrium correspondence* $\mathbf{F}^{-1}$ *contains a unique path that connects the equilibrium at* $\delta = 0$ *with the equilibrium at* $\delta = 1$. *In addition,* $\mathbf{F}^{-1}$ *may contain (one or more) loops that are disjoint from this "main path" and from each other.*




Limiting expected Herfindahl index $H^{\infty}$. Equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) < 1$ (solid line) and equilibria with $\varrho\left(\frac{\partial \mathbf{G}(\mathbf{x}(s))}{\partial \mathbf{x}}\Big|_{\delta(s)}\right) \geq 1$ (dotted line).

## Alternative Approaches to Computing all Equilibria

- If the system of equations is polynomial, then...
  - Judd, K., Renner, P. & Schmedders, K. (2012) "Finding all Pure-Strategy Equilibria in Games with Continuous Strategies."
  - Kubler, F, Schmedders, K. & Renner, P. (2013) "Computing all Solutions to Polynomial Equations in Economics."

- If movements though the state space are undirectional, then...
  - Judd, K. & Schmedders, K. (2004) "A Computational Approach to Proving Uniqueness in Dynamic Games."
  - Judd, K., Schmedders, K. & Yeltekin, S. (2012) "Optimal Rules for Patent Races."
  - Iskhakov, F., Rust, J. & Schjerning, B. (2014) "Recursive Lexicographical Search: Finding all Markov Perfect Equilibria of Finite State Directional Dynamic Games."
  - Iskhakov, F., Rust, J. & Schjerning, B. (2014) "The Dynamics of Bertrand Price Competition with Cost-Reducing Investments."